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# 2-Cosemisimplicial objects in a 2-category, permutohedra and deformations of pseudofunctors

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## Abstract

In this paper we take up again the deformation theory for  $K$ -linear pseudofunctors initiated in Elgueta (Adv. Math. 182 (2004) 204–277). We start by introducing a notion of 2-cosemisimplicial object in an arbitrary 2-category and analyzing the corresponding coherence question, where the permutohedra make their appearance. We then describe a general method to obtain usual cochain complexes of  $K$ -modules from (enhanced) 2-cosemisimplicial objects in the 2-category  $\mathbf{Cat}_K$  of small  $K$ -linear categories and prove that the deformation complex  $X^\bullet(\mathcal{F})$  introduced in Elgueta (to appear) can be obtained by this method from a 2-cosemisimplicial object that can be associated to  $\mathcal{F}$ . Finally, using this 2-cosemisimplicial object of  $\mathcal{F}$  and a generalization to the context of  $K$ -linear categories of the deviation calculus introduced by Markl and Stasheff for  $K$ -modules (J. Algebra 170 (1994) 122), it is shown that the obstructions to the integrability of an  $n$ th-order deformation of  $\mathcal{F}$  indeed correspond to cocycles in the third cohomology group  $H^3(X^\bullet(\mathcal{F}))$ , a question which remained open in Elgueta (Adv. Math. 182 (2004) 204–277).

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## 1. Introduction

In [6], we introduced a deformation complex for  $K$ -linear unitary pseudofunctors which turned out to describe the so-called purely pseudofunctorial first-order

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deformations. This was a generalization to the many objects setting of Yetter's deformation theory for monoidal functors (see [17,18]). A common feature of both deformation theories, which also appears in other categorical or 2-categorical deformation theories, such as Crane and Yetter's deformation theory for semigroupal categories [4] or the deformation theory for semigroupal 2-categories [6], is the presence of suitable "padding operators" in the definition of the coboundary maps. These operators may look like something artificial in the construction. One of the purposes of this paper is to give a framework where they appear most naturally. Our point of view is that the presence of such padding operators is a consequence of the intrinsically higher-dimensional nature of the structures that are being deformed. Conjecturally, they are the shadow of a higher-dimensional description, still to be found, of the corresponding deformation theory. In this sense, we guess that the right setting for studying categorical deformations should involve a suitable notion of 2-cochain complex, together with the corresponding notion of 2-co(semi)simplicial object in a 2-category. Along these lines, we introduce in this paper a notion of 2-co(semi)simplicial object in an arbitrary 2-category (a 2-dimensional version of the classical cosemisimplicial objects in a category), and we show that the deformation complex of a  $K$ -linear unitary pseudofunctor  $\mathcal{F}$  can be obtained from such an object that may be associated to  $\mathcal{F}$ . It is precisely in this process of going from the 2-cosemisimplicial object to the cochain complex that the padding operators appear. Presumably, this process involves a loss of information. It is then tempting to think that more information should be contained in the hypothetical *2-cochain complex* that should be derived from the 2-cosemisimplicial object, and that this 2-cochain complex could give a more complete description of the deformations of the pseudofunctor (including, for example, deformations at the level of 1-morphisms). At this point, it is worth mentioning the works by Street on cohomology with coefficients in an  $(n-)$ category [13–15]. This author has recently given (see [15]) a precise definition of what he calls the *descent  $n$ -category* of any cosimplicial  $n$ -category  $\mathcal{C}^\bullet$ . It seems possible that this notion of descent  $n$ -categories (or some variant of it) provides the right setting we are claiming for to give the cohomological description of the deformations of higher-dimensional algebraic structures.

As in any categorification process, in defining the notion of 2-cosemisimplicial object in a 2-category, suitable coherence conditions are introduced and the corresponding coherence theorem should be proved. In doing this, it turns out that the *permutohedra*, first introduced by Milgram in the context of iterated loop spaces [12], are the right family of convex polytopes describing the higher-order cosemisimplicial identities, in a way analogous to that encountered when weakening the associativity equation, where the role is played by the famous Stasheff associahedra.

The last purpose of the paper concerns higher-order obstructions. It remained as an open question in [6] if the obstructions to the integrability of an  $n$ th-order deformation indeed live in one of the cohomology groups, a condition which, according to Gerstenhaber [7], must satisfy any good cohomological deformation theory. We prove that this is indeed the case. More explicitly, we show that the obstructions correspond to 3-cocycles in the deformation complex introduced in [6]. To prove this, we use a generalization to the context of  $K$ -linear categories of the Markl and Stasheff deviation

calculus [11]. As it will be seen, the previously constructed 2-cosemisimplicial object turns out to be quite useful in making the proof easy to write.

The paper is organized as follows. Section 2 contains some definitions and preliminary results needed later. In Section 3, we recall the notion of deformation of a pseudofunctor we work with as well as the definition of the deformation complex as given in [6]. In Section 4 we define 2-cosemisimplicial objects in an arbitrary (strict) 2-category and prove the corresponding coherence theorem. In Section 5 we focus the attention on the special case of the 2-category  $\mathbf{Cat}_K$  of (small)  $K$ -linear categories and show that in this case usual cochain complexes of  $K$ -modules can be obtained from a suitably *enhanced* 2-cosemisimplicial object in  $\mathbf{Cat}_K$ . In the next section, we go back to the deformation theory of a pseudofunctor, proving that one can construct a (trivially enhanced) 2-cosemisimplicial object from any pseudofunctor and that, when the pseudofunctor is  $K$ -linear, its deformation complex coincides with one of the cochain complexes one may obtain by the method in the previous section. Finally, in Section 7 we generalize Markl and Stasheff deviation calculus to the context of arbitrary  $K$ -linear categories. This technique is used in the next section to prove that the obstructions to the integrability of a partial deformation indeed live in the corresponding cohomology.

## 2. Preliminaries

Unless otherwise indicated,  $K$  denotes a given commutative field. Let us first recall the definition of a pseudofunctor between 2-categories (see, for example, [1]).

**Definition 2.1.** If  $\mathcal{C}$  and  $\mathcal{D}$  are two 2-categories, a pseudofunctor from  $\mathcal{C}$  to  $\mathcal{D}$  is any quadruple  $\mathcal{F} = (|\mathcal{F}|, \mathcal{F}_*, \widehat{\mathcal{F}}_*, \mathcal{F}_0)$ , where

- $|\mathcal{F}| : |\mathcal{C}| \rightarrow |\mathcal{D}|$  is an object map;
- $\mathcal{F}_* = \{\mathcal{F}_{X,Y} : \mathcal{C}(X,Y) \rightarrow \mathcal{D}(\mathcal{F}(X), \mathcal{F}(Y))\}$  is a collection of functors, indexed by ordered pairs of objects  $X, Y \in |\mathcal{C}|$ ;
- $\widehat{\mathcal{F}}_* = \{\widehat{\mathcal{F}}_{X,Y,Z} : c_{\mathcal{F}(X), \mathcal{F}(Y), \mathcal{F}(Z)}^{\mathcal{D}} \circ (\mathcal{F}_{X,Y} \times \mathcal{F}_{Y,Z}) \Rightarrow \mathcal{F}_{X,Z} \circ c_{X,Y,Z}^{\mathcal{C}}\}$  is a collection of natural isomorphisms, indexed by ordered triples of objects  $X, Y, Z \in |\mathcal{C}|$  (the  $c_{-, -, -}^{\mathcal{C}}$  denote the composition functors in the 2-category  $\mathcal{C}$  and similarly  $c_{-, -, -}^{\mathcal{D}}$ ). Explicitly, this means having a 2-isomorphism<sup>1</sup>

$$\widehat{\mathcal{F}}_{X,Y,Z}(f, g) : \mathcal{F}_{Y,Z}(g) \circ \mathcal{F}_{X,Y}(f) \Rightarrow \mathcal{F}_{X,Z}(g \circ f)$$

for any path of 1-morphisms  $X \xrightarrow{f} Y \xrightarrow{g} Z$ , natural in  $(f, g)$ , and

- $\mathcal{F}_0 = \{\mathcal{F}_0(X) : \mathcal{F}_{X,X}(\text{id}_X) \Rightarrow \text{id}_{\mathcal{F}(X)}\}$  is a collection of 2-isomorphisms, indexed by objects  $X \in |\mathcal{C}|$ .

<sup>1</sup> In this paper, the arguments in  $\widehat{\mathcal{F}}$  are written in the reverse order to that used in [6].

Moreover, this data must satisfy the following coherence axioms (for short, the indexing objects are omitted so that we just write  $\widehat{\mathcal{F}}(f, g)$  and  $\mathcal{F}(f)$ ):

(A1) (*Composition axiom*) For all paths of 1-morphisms  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T$ , the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(h) \circ \mathcal{F}(g) \circ \mathcal{F}(f) & \xrightarrow{1_{\mathcal{F}(h)} \circ \widehat{\mathcal{F}}(f, g)} & \mathcal{F}(h) \circ \mathcal{F}(g \circ f) \\ \widehat{\mathcal{F}}(g, h) \circ 1_{\mathcal{F}(f)} \downarrow & & \downarrow \widehat{\mathcal{F}}(h, g \circ f) \\ \mathcal{F}(h \circ g) \circ \mathcal{F}(f) & \xrightarrow{\widehat{\mathcal{F}}(f, h \circ g)} & \mathcal{F}(h \circ g \circ f) \end{array}$$

(A2) (*Unit axioms*) For any 1-morphism  $f : X \rightarrow Y$ , the following equalities hold:

$$\begin{aligned} \widehat{\mathcal{F}}(\text{id}_X, f) &= 1_{\mathcal{F}(f)} \circ \mathcal{F}_0(X), \\ \widehat{\mathcal{F}}(\text{id}_Y, f) &= \mathcal{F}_0(Y) \circ 1_{\mathcal{F}(f)}. \end{aligned}$$

The whole set of 2-isomorphisms  $\widehat{\mathcal{F}}(f, g)$  and  $\mathcal{F}_0(X)$ , for all objects  $X$  and composable 1-morphisms  $f, g$ , will be called the *pseudofunctorial structure* on  $\mathcal{F}$ . When they are all identities the pseudofunctor is called a *2-functor*. When only the  $\mathcal{F}_0(X)$  are identities, we will call it a *unitary pseudofunctor*.

For later use, we give in the next lemma a “component-free” description of the above composition axiom. The proof is an easy exercise left to the reader.

**Lemma 2.2.** Let  $\mathcal{F} = (|\mathcal{F}|, \mathcal{F}_*, \widehat{\mathcal{F}}_*, \mathcal{F}_0)$  be the data defining a pseudofunctor between two 2-categories  $\mathfrak{C}$  and  $\mathfrak{D}$ , and let us define families of functors<sup>2</sup>  $\{\mathcal{F}_{X,Y,Z,T}^{(1,1,1)}\}$ ,  $\{\mathcal{F}_{X,Y,Z,T}^{(1,2)}\}$ ,  $\{\mathcal{F}_{X,Y,Z,T}^{(2,1)}\}$  and  $\{\mathcal{F}_{X,Y,Z,T}^{(3)}\}$  and natural isomorphisms  $\{\sigma_{X,Y,Z,T}^{12}\}$ ,  $\{\sigma_{X,Y,Z,T}^{24}\}$ ,  $\{\sigma_{X,Y,Z,T}^{13}\}$  and  $\{\sigma_{X,Y,Z,T}^{34}\}$ , both indexed by ordered quadruples  $(X, Y, Z, T)$  of objects in  $\mathfrak{C}$ , and, respectively, given by

$$\begin{aligned} \mathcal{F}_{X,Y,Z,T}^{(1,1,1)} &= c_{\mathcal{F}(X), \mathcal{F}(Z), \mathcal{F}(T)}^{\mathfrak{D}} \circ (c_{\mathcal{F}(X), \mathcal{F}(Y), \mathcal{F}(Z)}^{\mathfrak{D}} \times \text{id}_{\mathfrak{D}(\mathcal{F}(Z), \mathcal{F}(T))}) \\ &\quad \circ (\mathcal{F}_{X,Y} \times \mathcal{F}_{Y,Z} \times \mathcal{F}_{Z,T}), \end{aligned} \quad (2.1)$$

$$\mathcal{F}_{X,Y,Z,T}^{(1,2)} = c_{\mathcal{F}(X), \mathcal{F}(Y), \mathcal{F}(T)}^{\mathfrak{D}} \circ (\mathcal{F}_{X,Y} \times \mathcal{F}_{Y,T}) \circ (\text{id}_{\mathfrak{C}(X,Y)} \times c_{Y,Z,T}^{\mathfrak{C}}), \quad (2.2)$$

$$\mathcal{F}_{X,Y,Z,T}^{(2,1)} = c_{\mathcal{F}(X), \mathcal{F}(Z), \mathcal{F}(T)}^{\mathfrak{D}} \circ (\mathcal{F}_{X,Z} \times \mathcal{F}_{Z,T}) \circ (c_{X,Y,Z}^{\mathfrak{C}} \times \text{id}_{\mathfrak{C}(Z,T)}), \quad (2.3)$$

$$\mathcal{F}_{X,Y,Z,T}^{(3)} = \mathcal{F}_{X,T} \circ c_{X,Z,T}^{\mathfrak{C}} \circ (c_{X,Y,Z}^{\mathfrak{C}} \times \text{id}_{\mathfrak{C}(Z,T)}) \quad (2.4)$$

and<sup>3</sup>

$$\sigma_{X,Y,Z,T}^{12} = \mathbf{1}_{c_{\mathcal{F}(X), \mathcal{F}(Z), \mathcal{F}(T)}^{\mathfrak{D}}} \circ (\widehat{\mathcal{F}}_{X,Y,Z} \times \mathbf{1}_{\mathcal{F}_{Z,T}}), \quad (2.5)$$

<sup>2</sup> The meaning of the notation used to distinguish these families will be seen in Section 6.

<sup>3</sup> In this paper, identity 2-morphisms are generically denoted by  $1_f$ . But when the 1-morphism  $f$  is a functor we use a boldface 1, to emphasize the fact that it is an identity natural transformation.

$$\sigma_{X,Y,Z,T}^{24} = \widehat{\mathcal{F}}_{X,Z,T} \circ \mathbf{1}_{c_{X,Y,Z}^{\mathcal{E}} \times \text{id}_{\mathcal{E}(Z,T)}}, \quad (2.6)$$

$$\sigma_{X,Y,Z,T}^{13} = \mathbf{1}_{c_{\widehat{\mathcal{F}}(X),\widehat{\mathcal{F}}(Y),\widehat{\mathcal{F}}(T)}} \circ (\mathbf{1}_{\mathcal{F}_{X,Y}} \times \widehat{\mathcal{F}}_{Y,Z,T}), \quad (2.7)$$

$$\sigma_{X,Y,Z,T}^{34} = \widehat{\mathcal{F}}_{X,Z,T} \circ \mathbf{1}_{\text{id}_{\mathcal{E}(X,Y)} \times c_{Y,Z,T}^{\mathcal{E}}}. \quad (2.8)$$

Then, the previous composition axiom is equivalent to the commutativity of the diagrams of natural transformations

$$\begin{array}{ccc} \mathcal{F}_{X,Y,Z,T}^{(1,1,1)} & \xrightarrow{\sigma_{X,Y,Z,T}^{12}} & \mathcal{F}_{X,Y,Z,T}^{(2,1)} \\ \sigma_{X,Y,Z,T}^{13} \downarrow & & \downarrow \sigma_{X,Y,Z,T}^{24} \\ \mathcal{F}_{X,Y,Z,T}^{(1,2)} & \xrightarrow{\sigma_{X,Y,Z,T}^{34}} & \mathcal{F}_{X,Y,Z,T}^{(3)} \end{array} \quad (2.9)$$

for all ordered quadruples  $(X, Y, Z, T)$  of objects in  $\mathcal{E}$ .

The above definitions may be extended to the  $K$ -linear context using the Deligne product between  $K$ -linear categories and functors (see, for example, [18], Chapter 10). Furthermore, we will need to define the  $K[[h]]$ -linear extensions of the corresponding  $K$ -linear versions. Such definitions already appear in [6], although they were formulated without using the notion of Deligne product.

Recall that by a  $K$ -linear category one means a category  $\mathcal{C}$  enriched over the monoidal category  $\mathbf{Vect}_K$  of  $K$ -vector spaces. The corresponding topological version will be called a *complete  $K[[h]]$ -linear category*. By definition, it is a category  $\mathcal{C}$  enriched over the monoidal category  $\mathbf{K}[[h]]\text{-Mod}_c$  of separated and complete  $K[[h]]$ -modules.

**Definition 2.3.** A  $K$ -linear 2-category is a 2-category  $\mathcal{E}$  whose hom-categories  $\mathcal{E}(X, Y)$ , for all objects  $X, Y$  of  $\mathcal{E}$ , are  $K$ -linear, and whose composition functors  $c_{X,Y,Z}^{\mathcal{E}} : \mathcal{E}(X, Y) \times \mathcal{E}(Y, Z) \rightarrow \mathcal{E}(X, Z)$ , for all  $X, Y, Z$ , are  $K$ -bilinear or, equivalently,  $K$ -linear functors  $c_{X,Y,Z}^{\mathcal{E}} : \mathcal{E}(X, Y) \odot \mathcal{E}(Y, Z) \rightarrow \mathcal{E}(X, Z)$ , where  $\odot$  denotes the Deligne product of  $K$ -linear categories.

Similarly, a complete  $K[[h]]$ -linear 2-category is a 2-category  $\mathcal{E}$  whose hom-categories  $\mathcal{E}(X, Y)$ , for all objects  $X, Y$  of  $\mathcal{E}$ , are complete  $K[[h]]$ -linear categories and whose composition functors  $c_{X,Y,Z}^{\mathcal{E}} : \mathcal{E}(X, Y) \times \mathcal{E}(Y, Z) \rightarrow \mathcal{E}(X, Z)$ , for all  $X, Y, Z$ , are  $K[[h]]$ -bilinear or, equivalently,  $K[[h]]$ -linear functors  $c_{X,Y,Z}^{\mathcal{E}} : \mathcal{E}(X, Y) \widehat{\odot} \mathcal{E}(Y, Z) \rightarrow \mathcal{E}(X, Z)$ , where  $\widehat{\odot}$  denotes the topological Deligne product of complete  $K[[h]]$ -linear categories.

**Definition 2.4.** Given two  $K$ -linear 2-categories  $\mathcal{E}, \mathcal{D}$ , a  $K$ -linear pseudofunctor from  $\mathcal{E}$  to  $\mathcal{D}$  is a pseudofunctor  $\mathcal{F} : \mathcal{E} \rightarrow \mathcal{D}$  whose defining functors  $\mathcal{F}_{X,Y} : \mathcal{E}(X, Y) \rightarrow \mathcal{D}(F(X), F(Y))$ , for all objects  $X, Y$  of  $\mathcal{E}$ , are  $K$ -linear.

Similarly, by replacing the term  $K$ -linear by (complete)  $K[[h]]$ -linear, one gets the definition of  $K[[h]]$ -linear pseudofunctor between complete  $K[[h]]$ -linear 2-categories.

Notice that the defining natural isomorphisms  $\widehat{\mathcal{F}}_{X,Y,Z} : c_{\mathcal{F}(X),\mathcal{F}(Y),\mathcal{F}(Z)}^{\mathcal{D}} \circ (\mathcal{F}_{X,Y} \times \mathcal{F}_{Y,Z}) \Rightarrow \mathcal{F}_{X,Z} \circ c_{X,Y,Z}^{\mathcal{C}}$  of a  $K$ -linear pseudofunctor  $\mathcal{F}$  may also be considered as natural transformations  $\widehat{\mathcal{F}}_{X,Y,Z} : c_{\mathcal{F}(X),\mathcal{F}(Y),\mathcal{F}(Z)}^{\mathcal{D}} \circ (\mathcal{F}_{X,Y} \odot \mathcal{F}_{Y,Z}) \Rightarrow \mathcal{F}_{X,Z} \circ c_{X,Y,Z}^{\mathcal{C}}$ . The same thing is true for a  $K[[h]]$ -linear pseudofunctor, with the topological Deligne product  $\widehat{\odot}$  replaced by  $\odot$ . The reader may easily check that there is also an analog of Lemma 2.2 for  $K$ -linear and complete  $K[[h]]$ -linear pseudofunctors, where the cartesian product  $\times$  in the definition of functors (2.1)–(2.4) and natural transformations (2.5)–(2.8) must be replaced by the Deligne product  $\odot$  and the topological Deligne product  $\widehat{\odot}$ , respectively.

We will be mostly concerned with the  $K[[h]]$ -linear extensions of a  $K$ -linear 2-category or pseudofunctor. Let us first recall the definitions in the context of categories.

Given a  $K$ -linear category  $\mathcal{C}$ , its  $K[[h]]$ -linear extension, denoted by  $\mathcal{C}[[h]]$ , is the complete  $K[[h]]$ -linear category with the same objects as  $\mathcal{C}$  and  $K[[h]]$ -modules of morphisms given by

$$\mathcal{C}_h(X, Y) := \mathcal{C}(X, Y)[[h]], \quad X, Y \in |\mathcal{C}|,$$

where  $A[[h]]$ , for any  $K$ -module  $A$ , denotes the topologically free  $K[[h]]$ -module of formal power series in  $h$  with coefficients in  $A$ . Composition in  $\mathcal{C}[[h]]$  is defined in the obvious way in terms of the composition in  $\mathcal{C}$  and the product rule of formal power series. In particular, the identity morphisms in  $\mathcal{C}[[h]]$  are the same as in  $\mathcal{C}$ . It seems that these categories were introduced for the first time by Drinfeld [5] in his study of the quasiHopf algebras, providing the setting for the deformation theory of monoidal categories (see [4,18]). For its later use, let us state the following result, whose proof is left to the reader (it is the analog in the context of categories of a well-known result about the topological tensor product between topologically free  $K[[h]]$ -modules):

**Lemma 2.5.** *For any  $K$ -linear categories  $\mathcal{C}, \mathcal{D}$  there is an isomorphism of complete  $K[[h]]$ -linear categories  $\Psi_{\mathcal{C},\mathcal{D}} : \mathcal{C}[[h]] \widehat{\odot} \mathcal{D}[[h]] \xrightarrow{\cong} (\mathcal{C} \odot \mathcal{D})[[h]]$ .*

Given a  $K$ -linear functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between  $K$ -linear categories, its  $K[[h]]$ -linear extension, denoted by  $F[[h]]$ , is the  $K[[h]]$ -linear functor  $F[[h]] : \mathcal{C}[[h]] \rightarrow \mathcal{D}[[h]]$  acting on objects as  $F$  and such that

$$F[[h]] \left( \sum_{k \geq 0} f_k h^k \right) = \sum_{k \geq 0} F(f_k) h^k.$$

It is easy to check that  $(F' \circ F)[[h]] = F'[[h]] \circ F[[h]]$  and  $(\text{id}_{\mathcal{C}})[[h]] = \text{id}_{\mathcal{C}[[h]]}$  for all composable  $K$ -linear functors  $F, F'$  and  $K$ -linear categories  $\mathcal{C}$ . The proof of the next lemma is also left to the reader.

**Lemma 2.6.** *For any  $K$ -linear categories  $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_1, \mathcal{D}_2$  and  $K$ -linear functors  $F_i : \mathcal{C}_i \rightarrow \mathcal{D}_i$ ,  $i = 1, 2$ , we have*

$$(F_1 \odot F_2)[[h]] \circ \Psi_{\mathcal{C}_1, \mathcal{C}_2} = \Psi_{\mathcal{D}_1, \mathcal{D}_2} \circ (F_1[[h]] \widehat{\odot} F_2[[h]]).$$

Another easy but important fact needed later is the following:

**Lemma 2.7.** *For any  $K$ -linear functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  between arbitrary  $K$ -linear categories  $\mathcal{C}, \mathcal{D}$ , there is an isomorphism of  $K[[h]]$ -modules*

$$\text{Nat}(F, G)[[h]] \cong \text{Nat}(F[[h]], G[[h]])$$

*sending the formal power series  $\sum_{k \geq 0} \tau_k h^k$  to the natural transformation  $\tau_h : F[[h]] \Rightarrow G[[h]]$  with components*

$$\left( \sum_{k \geq 0} \tau_k h^k \right)_X = \sum_{k \geq 0} (\tau_k)_X h^k, \quad X \in |\mathcal{C}|.$$

*Furthermore, under this identification, the vertical and horizontal compositions of natural transformations are given by the usual product rule of formal power series.*

**Proof.** By definition, a natural transformation  $\tau_h : F[[h]] \Rightarrow G[[h]]$  involves a collection of morphisms  $(\tau_h)_X : F(X) \rightarrow G(X)$  in  $\mathcal{D}[[h]]$ , for all objects  $X$  of  $\mathcal{C}$ . But a generic such morphism is of the form

$$(\tau_h)_X = \sum_{n \geq 0} (\tau_n)_X h^n.$$

The proof reduces to show that the naturality of  $(\tau_h)_X$  in  $X$  is equivalent to the naturality in  $X$  of the  $(\tau_n)_X$ , for all  $n \geq 0$ . This last condition may be shown by an easy induction which is left to the reader. As regards the formula for the vertical composition, it immediately follows from the definition of composition in  $\mathcal{D}[[h]]$ .  $\square$

The corresponding notions of  $K[[h]]$ -linear extension in the 2-category setting can now be formulated as follows.

**Definition 2.8.** Let  $\mathcal{C}$  be a  $K$ -linear 2-category. Then, its  $K[[h]]$ -linear extension is the complete  $K[[h]]$ -linear 2-category  $\mathcal{C}[[h]]$  given by the following data:

- (i) The objects of  $\mathcal{C}[[h]]$  are the same as in  $\mathcal{C}$ .
- (ii) The hom-categories  $\mathcal{C}[[h]](X, Y)$  are the  $K[[h]]$ -linear extensions of the corresponding categories, i.e., for all objects  $X, Y$ ,

$$\mathcal{C}[[h]](X, Y) := \mathcal{C}(X, Y)[[h]]. \quad (2.10)$$

- (iii) The composition functors  $c_{X, Y, Z}^{\mathcal{C}[[h]]} : \mathcal{C}[[h]](X, Y) \hat{\otimes} \mathcal{C}[[h]](Y, Z) \rightarrow \mathcal{C}[[h]](X, Z)$ , for any objects  $X, Y, Z$  of  $\mathcal{C}$ , are given by

$$c_{X, Y, Z}^{\mathcal{C}[[h]]} := c_{X, Y, Z}^{\mathcal{C}}[[h]] \circ \Psi_{X, Y, Z}, \quad (2.11)$$

where  $c_{X, Y, Z}^{\mathcal{C}}[[h]]$  is the  $K[[h]]$ -linear extension of the composition functor  $c_{X, Y, Z}^{\mathcal{C}}$  of  $\mathcal{C}$  and  $\Psi_{X, Y, Z} = \Psi_{\mathcal{C}(X, Y), \mathcal{C}(Y, Z)}$  (see Lemma 2.5).

- (iv) The identity 1-morphisms  $\text{id}_X$  are the same as in  $\mathcal{C}$ .

The reader may easily check that the above data indeed defines a  $(K[[h]]$ -linear) 2-category. Notice that according to (2.10), the 1-morphisms in  $\mathcal{C}[[h]]$  are exactly the

same as in  $\mathfrak{C}$  but a generic 2-morphism  $\tau_h : f \Rightarrow f'$  between two such 1-morphisms  $f, f' : X \rightarrow Y$  is of the form of a formal power series

$$\tau_h = \tau_0 + \tau_1 h + \tau_2 h^2 + \cdots$$

with the  $\tau_i : f \Rightarrow f'$ ,  $i \geq 0$ , 2-morphisms in  $\mathfrak{C}$ . Also implicit in (2.10) is the fact that the vertical composition of two such 2-morphisms is given by the usual product rule of formal power series, while (2.11) means that the composition of 1-morphisms in  $\mathfrak{C}[[h]]$  is the same as in  $\mathfrak{C}$  and the horizontal composition of two 2-morphisms  $\tau_h : f \Rightarrow f' : X \rightarrow Y$  and  $\sigma_h : g \Rightarrow g' : Y \rightarrow Z$  is given by the product rule.

Before giving the corresponding notion of  $K[[h]]$ -linear extension for  $K$ -linear pseudofunctors, let us first remark that for any  $K$ -linear pseudofunctor  $\mathcal{F} : \mathfrak{C} \rightarrow \mathfrak{D}$ , we have (see Lemma 2.6)

$$\begin{aligned} c_{\mathcal{F}(X), \mathcal{F}(Y), \mathcal{F}(Z)}^{\mathfrak{D}[[h]]} \circ (\mathcal{F}_{X,Y}[[h]] \hat{\odot} \mathcal{F}_{Y,Z}[[h]]) \\ = (c_{\mathcal{F}(X), \mathcal{F}(Y), \mathcal{F}(Z)}^{\mathfrak{D}} \circ (\mathcal{F}_{X,Y} \odot \mathcal{F}_{Y,Z}))[[h]] \circ \Psi_{X,Y,Z} \end{aligned}$$

and

$$\mathcal{F}_{X,Z}[[h]] \circ c_{X,Y,Z}^{\mathfrak{C}[[h]]} = (\mathcal{F}_{X,Z} \circ c_{X,Y,Z}^{\mathfrak{C}})[[h]] \circ \Psi_{X,Y,Z}.$$

Hence, the following definition makes sense (see also Lemma 2.7).

**Definition 2.9.** Let  $\mathcal{F} : \mathfrak{C} \rightarrow \mathfrak{D}$  be a  $K$ -linear pseudofunctor between  $K$ -linear 2-categories. Then, the  $K[[h]]$ -linear extension of  $\mathcal{F}$  is the  $K[[h]]$ -linear pseudofunctor  $\mathcal{F}[[h]] : \mathfrak{C}[[h]] \rightarrow \mathfrak{D}[[h]]$  acting on objects as  $\mathcal{F}$  and whose remaining structural data is given by:

- (i)  $\mathcal{F}[[h]]_{X,Y} = \mathcal{F}_{X,Y}[[h]]$  (the  $K[[h]]$ -linear extension of  $\mathcal{F}_{X,Y}$ ).
- (ii)  $\mathcal{F}[[h]]_{X,Y,Z} = \widehat{\mathcal{F}}_{X,Y,Z} \circ \mathbf{1}_{\Psi_{X,Y,Z}}$  (here,  $\widehat{\mathcal{F}}_{X,Y,Z}$  stands for a formal power series of natural transformation with only zero-order term).
- (iii)  $\mathcal{F}[[h]]_0(X) = \mathcal{F}_0(X)$ .

We leave to the reader to check that the previous data indeed define a  $K[[h]]$ -linear pseudofunctor between  $\mathfrak{C}[[h]]$  and  $\mathfrak{D}[[h]]$ . Notice that, according to conditions (i) and (ii), for any path  $X \xrightarrow{f} Y \xrightarrow{g} Z$  and any 2-morphism  $\tau_h = \tau_0 + \tau_1 h + \cdots : f \Rightarrow f'$  in  $\mathfrak{C}[[h]]$  we have

$$\begin{aligned} \mathcal{F}[[h]](f) &= \mathcal{F}(f), \\ \mathcal{F}[[h]](\tau_h) &= \mathcal{F}(\tau_0) + \mathcal{F}(\tau_1)h + \cdots, \\ \widehat{\mathcal{F}[[h]]}(f, g) &= \widehat{\mathcal{F}}(f, g). \end{aligned}$$

### 3. Deformation complex of a $K$ -linear pseudofunctor

Given a  $K$ -linear pseudofunctor  $\mathcal{F}$ , we introduced in [6] a cochain complex  $(X^\bullet(\mathcal{F}), \delta)$  which in the unitary case described the purely pseudofunctorial first-order deformations



of  $\mathcal{F}$ . A fundamental question which remained open was if the obstructions to the integrability of a partial deformation live in some of the cohomology groups. This point is settled down in Section 8 using an analog of Markl and Stasheff deviation calculus [11]. In this section, we recall the necessary definitions from [6].

**Definition 3.1.** Let  $\mathcal{C}, \mathcal{D}$  two  $K$ -linear 2-categories and  $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$  a  $K$ -linear pseudo-functor. Then, by a purely pseudofunctorial formal deformation of  $\mathcal{F}$  we mean any  $K[[h]]$ -linear pseudofunctor  $\mathcal{F}_h : \mathcal{C}[[h]] \rightarrow \mathcal{D}[[h]]$  differing from the  $K[[h]]$ -linear extension  $\mathcal{F}[[h]]$  (see Definition 2.9) only in the pseudofunctorial structure, which must be of the form

$$(\widehat{\mathcal{F}}_h)_{X,Y,Z} = \left( \sum_{k \geq 0} \widehat{\mathcal{F}}_{X,Y,Z}^k h^k \right) \circ \mathbf{1}_\Psi, \quad (3.1)$$

$$(\mathcal{F}_h)_0(X) = \sum_{k \geq 0} \mathcal{F}_0^k(X) h^k \quad (3.2)$$

with  $\widehat{\mathcal{F}}_{X,Y,Z}^0 = \widehat{\mathcal{F}}_{X,Y,Z}$  and  $\mathcal{F}_0^0(X) = \mathcal{F}_0(X)$  for all objects  $X, Y, Z$  of  $\mathcal{C}$ .

Notice that  $\mathcal{F}[[h]]$  itself gives an example of such a deformation, called the *null deformation*, where  $\widehat{\mathcal{F}}_{X,Y,Z}^k = 0$  and  $\mathcal{F}_0^k(X) = 0$  for all  $k \geq 1$ .

Clearly, a purely pseudofunctorial formal deformation of  $\mathcal{F}$  is completely given by the families of natural transformations  $\{\widehat{\mathcal{F}}_{X,Y,Z}^k\}_{X,Y,Z}$  and 2-morphisms  $\{\mathcal{F}_0^k(X)\}_X$ , for all  $k \geq 1$ . However, they are not arbitrary. They must be such that the corresponding natural transformations (3.1) and 2-morphisms (3.2) indeed define a pseudofunctorial structure on  $\mathcal{F}_h$ . Next result makes precise the conditions they must satisfy in a form suitable to our purposes. In particular, the diagrams which appear are of the right kind for the notion of deviation introduced in Section 7 to make sense.

**Lemma 3.2.** Let  $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$  be a  $K$ -linear pseudofunctor. Then, the families  $\{\widehat{\mathcal{F}}_{X,Y,Z}^k\}_{X,Y,Z}$  and  $\{\mathcal{F}_0^k(X)\}_X$ ,  $k \geq 1$ , define a purely pseudofunctorial formal deformation  $\mathcal{F}_h$  of  $\mathcal{F}$  if and only if

(1) For all objects  $X, Y, Z, T \in |\mathcal{C}|$ , the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_{X,Y,Z,T}^{(1,1)}[[h]] & \xrightarrow{\sigma_{X,Y,Z,T}^{12}(h)} & \mathcal{F}_{X,Y,Z,T}^{(2,1)}[[h]] \\ \sigma_{X,Y,Z,T}^{13}(h) \downarrow & & \downarrow \sigma_{X,Y,Z,T}^{24}(h) \\ \mathcal{F}_{X,Y,Z,T}^{(1,2)}[[h]] & \xrightarrow{\sigma_{X,Y,Z,T}^{34}(h)} & \mathcal{F}_{X,Y,Z,T}^{(3)}[[h]] \end{array} \quad (3.3)$$

where

$$\sigma_{X,Y,Z,T}^{12}(h) := \sum_{k \geq 0} [\mathbf{1}_{c_{\mathcal{F}(X), \mathcal{F}(Z), \mathcal{F}(T)}^{\mathcal{D}}} \circ (\widehat{\mathcal{F}}_{X,Y,Z}^k \odot \mathbf{1}_{\mathcal{F}_{Z,T}})] h^k, \quad (3.4)$$

$$\sigma_{X,Y,Z,T}^{24}(h) := \sum_{k \geq 0} [\widehat{\mathcal{F}}_{X,Z,T}^k \circ \mathbf{1}_{c_{X,Y,Z}^{\mathfrak{C}} \odot \text{id}_{\mathfrak{C}(Z,T)}}] h^k, \quad (3.5)$$

$$\sigma_{X,Y,Z,T}^{13}(h) := \sum_{k \geq 0} [\mathbf{1}_{c_{\mathfrak{F}(X), \mathfrak{F}(Y), \mathfrak{F}(T)}}^{\mathfrak{D}} \circ (\mathbf{1}_{\mathcal{F}_{X,Y}} \odot \widehat{\mathcal{F}}_{Y,Z,T}^k)] h^k, \quad (3.6)$$

$$\sigma_{X,Y,Z,T}^{34}(h) := \sum_{k \geq 0} [\widehat{\mathcal{F}}_{X,Z,T}^k \circ \mathbf{1}_{\text{id}_{\mathfrak{C}(X,Y)} \odot c_{Y,Z,T}^{\mathfrak{C}}}] h^k. \quad (3.7)$$

(2) For all objects  $X, Y \in |\mathfrak{C}|$ , all 1-morphisms  $f : X \rightarrow Y$  and all  $k \geq 1$ , the following equalities hold

$$\widehat{\mathcal{F}}_{X,X,Y}^k(\text{id}_X, f) = \mathbf{1}_{\mathcal{F}(f)} \circ \mathcal{F}_0^k(X), \quad (3.8)$$

$$\widehat{\mathcal{F}}_{X,Y,Y}^k(f, \text{id}_Y) = \mathcal{F}_0^k(Y) \circ \mathbf{1}_{\mathcal{F}(f)}. \quad (3.9)$$

The set of equations (3.3) together with (3.8) and (3.9) play the role of the associativity equation in the study of the formal deformations of an associative algebra [7], and are called the *structural* or *deformation equations*.

**Proof.** By the topological  $K[[h]]$ -linear version of Lemma 2.2, we know that the composition axiom is equivalent to the commutativity of the diagrams

$$\begin{array}{ccc} (\mathcal{F}_h)_{X,Y,Z,T}^{(1,1,1)} & \xrightarrow{(\sigma_h)_{X,Y,Z,T}^{12}} & (\mathcal{F}_h)_{X,Y,Z,T}^{(2,1)} \\ (\sigma_h)_{X,Y,Z,T}^{13} \downarrow & & \downarrow (\sigma_h)_{X,Y,Z,T}^{24} \\ (\mathcal{F}_h)_{X,Y,Z,T}^{(1,2)} & \xrightarrow{(\sigma_h)_{X,Y,Z,T}^{34}} & (\mathcal{F}_h)_{X,Y,Z,T}^{(3)} \end{array} \quad (3.10)$$

for all objects  $X, Y, Z, T \in |\mathfrak{C}|$ . Now, using Lemma 2.6, we obtain that

$$(\mathcal{F}_h)_{X,Y,Z,T}^{\alpha} = \mathcal{F}_{X,Y,Z,T}^{(\alpha)}[[h]] \circ \Psi_{X,Y,Z,T}$$

for all  $\alpha = (1, 1, 1), (2, 1), (1, 2), (3)$ , where  $\Psi_{X,Y,Z,T} = \Psi_{\mathfrak{C}(X,Y), \mathfrak{C}(Y,Z), \mathfrak{C}(Z,T)}$  denotes the canonical isomorphism  $\mathfrak{C}(X,Y)[[h]] \widehat{\odot} \mathfrak{C}(Y,Z)[[h]] \widehat{\odot} \mathfrak{C}(Z,T)[[h]] \cong (\mathfrak{C}(X,Y) \odot \mathfrak{C}(Y,Z) \odot \mathfrak{C}(Z,T))[[h]]$ , whose existence follows from Lemma 2.5. On the other hand, a straightforward computation shows that

$$(\sigma_h)_{X,Y,Z,T}^{ij} = \sigma_{X,Y,Z,T}^{ij}(h) \circ \mathbf{1}_{\Psi_{X,Y,Z,T}}$$

for all pairs  $i, j$ , where the  $\sigma_{X,Y,Z,T}^{ij}(h)$  are the natural transformations (3.4)–(3.7). Hence, condition (2.9) on  $\mathcal{F}_h$  takes the form

$$\begin{array}{ccc} \mathcal{F}_{X,Y,Z,T}^{(1,1,1)}[[h]] \circ \Psi_{X,Y,Z,T} & \xrightarrow{\sigma_{X,Y,Z,T}^{12}(h) \circ \mathbf{1}_{\Psi_{X,Y,Z,T}}} & \mathcal{F}_{X,Y,Z,T}^{(2,1)}[[h]] \circ \Psi_{X,Y,Z,T} \\ \sigma_{X,Y,Z,T}^{13}(h) \circ \mathbf{1}_{\Psi_{X,Y,Z,T}} \downarrow & & \downarrow \sigma_{X,Y,Z,T}^{24}(h) \circ \mathbf{1}_{\Psi_{X,Y,Z,T}} \\ \mathcal{F}_{X,Y,Z,T}^{(1,2)}[[h]] \circ \Psi_{X,Y,Z,T} & \xrightarrow{\sigma_{X,Y,Z,T}^{34}(h) \circ \mathbf{1}_{\Psi_{X,Y,Z,T}}} & \mathcal{F}_{X,Y,Z,T}^{(3)}[[h]] \circ \Psi_{X,Y,Z,T} \end{array}$$

By the interchange law this is equivalent to

$$(\sigma_{X,Y,Z,T}^{24}(h) \cdot \sigma_{X,Y,Z,T}^{12}(h)) \circ \mathbf{1}_{\Psi_{X,Y,Z,T}} = (\sigma_{X,Y,Z,T}^{34}(h) \cdot \sigma_{X,Y,Z,T}^{13}(h)) \circ \mathbf{1}_{\Psi_{X,Y,Z,T}}$$

and, since  $\Psi_{X,Y,Z,T}$  is an isomorphism (in particular, essentially surjective), the terms in  $\Psi_{X,Y,Z,T}$  may indeed be cancelled to give the equivalent condition (3.3). The proof that equalities (3.8) and (3.9) are in turn equivalent to the unit axioms on the deformed pseudofunctor  $\mathcal{F}_h$  is left to the reader.  $\square$

Together with the notion of purely pseudofunctorial *formal* deformation, in [6] we also introduced the corresponding notion of purely pseudofunctorial *n*th-order deformation, for all  $n \geq 1$ . It is defined in the same way as the formal deformations by replacing the ring of formal power series  $K[[h]]$  by the ring of truncated polynomials  $K[h]/(h^n)$ . Using arguments similar to those made above, it may be shown that such a deformation is completely given by families  $\{\widehat{\mathcal{F}}_{X,Y,Z}^k\}$  and  $\{\mathcal{F}_0^k(X)\}$  as above, for  $k = 1, \dots, n$ , satisfying the deformation equations (3.3) up to  $h^{n+1}$  and (3.8)–(3.9) for all  $k = 1, \dots, n$ . The details are left to the reader.

Then, for any  $K$ -linear pseudofunctor  $\mathcal{F} : \mathfrak{C} \rightarrow \mathfrak{D}$ , we defined in [6] a cochain complex  $X^\bullet(\mathcal{F})$  whose vector spaces  $X^n(\mathcal{F})$  were given by

$$X^n(\mathcal{F}) := \begin{cases} \prod_{(X_0, \dots, X_n) \in |\mathfrak{C}|^{n+1}} \text{Nat}(\mathcal{F}_{X_0, \dots, X_n}^{(1, \dots, 1)}, \mathcal{F}_{X_0, \dots, X_n}^{(n)}), & n \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\mathcal{F}_{X_0, \dots, X_n}^{(1, \dots, 1)} := c_{\mathcal{F}(X_0), \dots, \mathcal{F}(X_n)}^{\mathfrak{D}} \circ (\mathcal{F}_{X_0, X_1} \odot \mathcal{F}_{X_1, X_2} \odot \dots \odot \mathcal{F}_{X_{n-1}, X_n}),$$

$$\mathcal{F}_{X_0, \dots, X_n}^{(n)} := \mathcal{F}_{X_0, X_n} \circ c_{X_0, \dots, X_n}^{\mathfrak{C}}$$

for all  $n \geq 2$  (they are the components of two particular  $\mathcal{F}$ -iterates of multiplicity  $n$  chosen as references) and

$$\mathcal{F}_{X_0, X_1}^{(1)} := \mathcal{F}_{X_0, X_1}$$

if  $n = 1$  (the unique  $\mathcal{F}$ -iterate of multiplicity 1). Here, the  $c^{\mathfrak{C}}$  and  $c^{\mathfrak{D}}$  indexed by  $n + 1$  objects,  $n \geq 3$ , denote the unique  $n$ th-order induced composition functors in the corresponding 2-category. The coboundary map  $\delta : X^{n-1}(\mathcal{F}) \rightarrow X^n(\mathcal{F})$ ,  $n \geq 2$ , was then defined in terms of the “padding” operators  $\lceil - \rceil$  associated to  $\mathcal{F}$  (see [6]) by the formula

$$\begin{aligned} (\delta\phi)(f_0, f_1, \dots, f_{n-1}) &= \lceil 1_{\mathcal{F}(f_{n-1})} \circ \phi(f_0, \dots, f_{n-2}) \rceil_{\mathcal{F}(X_0), \mathcal{F}(X_n)} \\ &\quad + \sum_{i=1}^{n-1} (-1)^i \lceil \phi(f_0, \dots, f_i \circ f_{i-1}, \dots, f_{n-1}) \rceil_{\mathcal{F}(X_0), \mathcal{F}(X_n)} \\ &\quad + (-1)^n \lceil \phi(f_1, \dots, f_{n-1}) \circ 1_{\mathcal{F}(f_0)} \rceil_{\mathcal{F}(X_0), \mathcal{F}(X_n)} \end{aligned}$$

with  $\phi \in X^{n-1}(\mathcal{F})$  and  $f_i \in |\mathfrak{C}(X_i, X_{i+1})|$ ,  $i = 0, \dots, n - 1$  (notice that 1-morphisms  $f_i$  are indexed differently with respect to the notation in [6] and that, as arguments of  $\phi$ , they are written in the reverse order). We proved then the following:

**Theorem 3.3** (Elgueta [6]). *Let  $\mathcal{F}$  be a  $K$ -linear unitary pseudofunctor and let us denote by  $H^\bullet(\mathcal{F})$  the cohomology of the corresponding deformation complex as defined above. Then, the equivalence classes of the purely pseudofunctorial first order deformations are in one-one correspondence with the elements of  $H^2(X^\bullet(\mathcal{F}))$ .*

#### 4. 2-Cosemisimplicial objects in a 2-category

As mentioned in the introduction, in this section we introduce a notion of 2-cosemisimplicial object in a 2-category as a sort of categorification of the classical notion of cosemisimplicial object in a category (see, for example, [16]). Our original motivation for doing this was to see that, associated to any pseudofunctor between 2-categories, we have such an object, and that the cochain complex of a  $K$ -linear pseudofunctor in the previous section can be obtained from it. This is done in Section 6.

Recall that, given any category  $\mathcal{C}$ , a cosemisimplicial object in  $\mathcal{C}$  is any covariant functor  $K : \Delta_s \rightarrow \mathcal{C}$ , where  $\Delta_s$  (the semisimplicial category) is the subcategory of the simplicial category  $\Delta$  whose morphisms are the injections  $\alpha : [i] \hookrightarrow [n]$  (see [16]). To define the corresponding categorified notion,  $\mathcal{C}$  should be replaced by a bicategory  $\mathfrak{C}$ ,  $\Delta_s$  by a suitable ‘semisimplicial bicategory’  $2\Delta_s$  and  $K : \Delta_s \rightarrow \mathcal{C}$  by a pseudofunctor  $\mathcal{F} : 2\Delta_s \rightarrow \mathfrak{C}$ . The outstanding point is what we should take as semisimplicial bicategory  $2\Delta_s$ . A priori, the only reasonable condition we have on it is that it should be a categorification of  $\Delta_s$ . But the categorification of a given mathematical structure is not unique in general. For example, the set  $\mathbb{N}$  of natural numbers as a “rig” has the category of finite sets as well as the category of finite-dimensional vector spaces over a given field  $K$  as two nonequivalent categorifications, or the usual notion of commutative monoid, which has both the notions of symmetric monoidal category and braided monoidal category as two nonequivalent categorifications. To avoid making such a choice and at the same time to have a description as explicit as possible of the notion of 2-cosemisimplicial object, we will take as our starting point the definition of cosemisimplicial object in  $\mathcal{C}$  which follows from the presentation of  $\Delta_s$  in terms of generators and relations. Thus, using such presentation of  $\Delta_s$ , it can be shown that a cosemisimplicial object in  $\mathcal{C}$  is the same thing as a sequence of objects  $K_0, K_1, \dots$  in  $\mathcal{C}$  together with coface morphisms  $\partial_n^i : K^{n-1} \rightarrow K^n$ ,  $i = 0, \dots, n$ ,  $n \geq 1$ , satisfying the cosemisimplicial identities

$$\partial_{n+1}^j \circ \partial_n^i = \partial_{n+1}^i \circ \partial_n^{j-1}, \quad 0 \leq i < j \leq n+1. \quad (4.1)$$

We then take as definition in the 2-dimensional setting the following (to simplify, we further restrict to the context of 2-categories).

**Definition 4.1.** Given a 2-category  $\mathfrak{C}$ , a 2-cosemisimplicial object in  $\mathfrak{C}$  is any sequence of objects  $X^0, X^1, \dots$  in  $\mathfrak{C}$  together with 1-morphisms (the coface maps)  $\partial_n^i : X^{n-1} \rightarrow X^n$ , for all  $i = 0, \dots, n$  and  $n \geq 1$ , and 2-isomorphisms (the cosemisimplicial coherers)

$\tau_{ij}^n : \partial_{n+1}^j \circ \partial_n^i \Rightarrow \partial_{n+1}^i \circ \partial_n^{j-1}$ ,  $0 \leq i < j \leq n+1$ , such that the diagrams

$$\begin{array}{ccccc} \partial_{n+2}^k \circ \partial_{n+1}^j \circ \partial_n^i & \xrightarrow{1_{\partial_{n+2}^k} \circ \tau_{ij}^n} & \partial_{n+2}^k \circ \partial_{n+1}^i \circ \partial_n^{j-1} & \xrightarrow{\tau_{ik}^{n+1} \circ 1_{\partial_n^{j-1}}} & \partial_{n+2}^i \circ \partial_{n+1}^{k-1} \circ \partial_n^{j-1} \\ \Downarrow \tau_{jk}^{n+1} \circ 1_{\partial_n^i} & & & & 1_{\partial_{n+2}^i} \circ \tau_{j-1,k-1}^n \Downarrow \\ \partial_{n+2}^j \circ \partial_{n+1}^{k-1} \circ \partial_n^i & \xrightarrow{1_{\partial_{n+2}^j} \circ \tau_{ik}^n} & \partial_{n+2}^j \circ \partial_{n+1}^i \circ \partial_n^{k-2} & \xrightarrow{\tau_{ij}^{n+1} \circ 1_{\partial_n^{k-2}}} & \partial_{n+2}^i \circ \partial_{n+1}^{j-1} \circ \partial_n^{k-2} \end{array}$$

commute for all  $0 \leq i < j < k \leq n+2$  and all  $n \geq 1$ .

For short, such a 2-cosemisimplicial object will be denoted by the triple  $(X^\bullet, \partial, \tau)$  or just by  $X^\bullet$ , when there is no confusion. Notice that this definition includes as special cases the usual cosemisimplicial objects in a category  $\mathcal{C}$  when we think of  $\mathcal{C}$  as the 2-category with only the identity 2-morphisms.

The commutative diagrams in the above definition are the coherence laws that appear in any categorification process, and they are imposed to get the corresponding coherence theorem. To state this theorem, let us consider, for any  $s, k \geq 1$ , the subcategory  $\mathcal{C}_{s,k}$  of  $\mathfrak{C}(X^{s-1}, X^{s+k})$  with objects all composites of the coface maps, i.e., all 1-morphisms  $f : X^{s-1} \rightarrow X^{s+k}$  of the form

$$f = \partial_{s+k}^{i_k} \circ \partial_{s+k-1}^{i_{k-1}} \circ \cdots \circ \partial_s^{i_0}$$

for  $i_j = 0, 1, \dots, s+j$  ( $j = 0, \dots, k$ ). We will refer to such 1-morphisms as the  $\partial$ -paths from  $X^{s-1}$  to  $X^{s+k}$ . Given two such  $\partial$ -paths  $f, f'$ , the morphisms from  $f$  to  $f'$  in  $\mathcal{C}_{s,k}$  are all possible pastings of the coherers  $\tau_{ij}^n$ 's and the identity 2-morphisms of the coface maps giving a 2-morphism between them. They will be denoted by  $\sigma : f \Rightarrow f'$  because they are actually 2-morphisms in  $\mathfrak{C}$ . Thus, a generic morphism  $\sigma : f \Rightarrow f'$  in  $\mathcal{C}_{s,k}$  is of the form

$$\sigma = (1_{f'_1} \circ \tau_{i_1 j_1}^{n_1} \circ 1_{f_1}) \cdot (1_{f'_2} \circ \tau_{i_2 j_2}^{n_2} \circ 1_{f_2}) \cdot \cdots \cdot (1_{f'_q} \circ \tau_{i_q j_q}^{n_q} \circ 1_{f_q}),$$

for some  $\partial$ -paths  $f_\alpha, f'_\alpha$  and indices  $i_\alpha, j_\alpha, n_\alpha$ , with  $\alpha = 1, \dots, q$  (the dot denotes the vertical composition of 2-morphisms in  $\mathfrak{C}$ ). The 2-morphisms in  $\mathfrak{C}$  of the form  $1_{f'} \circ \tau_{ij}^n \circ 1_f$ , for  $f, f'$   $\partial$ -paths, will be called *expanded coherers*. For example, the composites  $\partial_3^3 \circ \partial_2^1 \circ \partial_1^0$  and  $\partial_3^0 \circ \partial_2^0 \circ \partial_1^1$  define two objects of  $\mathcal{C}_{1,2}$ , and a morphism in  $\mathcal{C}_{1,2}$  between them is given by the pasting

$$(\tau_{01}^2 \circ 1_{\partial_1^1}) \cdot (1_{\partial_3^1} \circ \tau_{02}^1) \cdot (\tau_{13}^2 \circ 1_{\partial_1^0}).$$

The coherence theorem states then the following:

**Theorem 4.2.** *Let  $s, k \geq 1$ . Then, for any two objects  $f, f'$  in  $\mathcal{C}_{s,k}$ , there is at most one morphism (actually, an isomorphism) in  $\mathcal{C}_{s,k}$  from  $f$  to  $f'$ .*

Such a unique isomorphism will be called the *canonical 2-isomorphism* from  $f$  to  $f'$ , to distinguish it from all other possible 2-morphisms between  $f$  and  $f'$  that may exist in  $\mathfrak{C}$ .

To prove the theorem, let us consider the graph  $G_{s,k}$  with vertices all  $\partial$ -paths  $f : X^{s-1} \rightarrow X^{s+k}$  and with edges all the expanded coherers (hence,  $\mathcal{C}_{s,k}$  is the quotient

of the free groupoid generated by  $G_{s,k}$  modulo the above coherence relations). It has  $(s+1)(s+2)\cdots(s+k+1)$  vertices and it is a degree  $k$  regular graph (i.e., for any vertex, the total number of incident edges is equal to  $k$ ). It follows that  $G_{s,k}$  has  $\frac{1}{2}k(s+1)\cdots(s+k+1)$  edges. Let us identify the vertex  $\partial_{s+k}^{i_k} \circ \partial_{s+k-1}^{i_{k-1}} \circ \cdots \circ \partial_s^{i_0}$  in  $G_{s,k}$  with the  $(k+1)$ -tuple  $(i_0, \dots, i_k)$ . The sum  $i_0 + \cdots + i_k$  will be called the *height* of the vertex and denoted by  $h(i_0, \dots, i_k)$ . We further define the *rank* of such a vertex, denoted by  $r(i_0, \dots, i_k)$ , as the number of strictly positive jumps we meet when going from  $i_0$  to  $i_k$ . Hence,  $0 \leq r(i_0, \dots, i_k) \leq k$ . For example,  $r(1, 2, 3, 2, 4) = 3$  and  $r(1, 1, 2, 3) = 2$ . If we agree that an edge goes *out* of a vertex when the vertex is the domain of the expanded coherer represented by that edge, while it goes *into* a vertex when the vertex is its codomain (equivalently, the domain of the inverse morphism), then the rank of a vertex corresponds to the number of edges going out of the vertex. A vertex  $(i_0, \dots, i_k)$  will be called an *out-vertex* when its rank is  $k$  (all edges go out of the vertex), and an *in-vertex* when its rank is zero (all edges go into the vertex). Note that the out-vertices in  $G_{s,k}$  are in one–one correspondence with the subsets of  $k+1$  elements of the set  $\{0, 1, \dots, s+k\}$ , because it must be  $i_0 < i_1 < \cdots < i_k$ . In particular, two different out-vertices have different heights. Finally, if the edges of a path in  $G_{s,k}$ , taken in order, involve only expanded coherers and none of its inverses (resp. only inverses of the expanded coherers), the path will be called *directed* (resp. *inversely directed*).

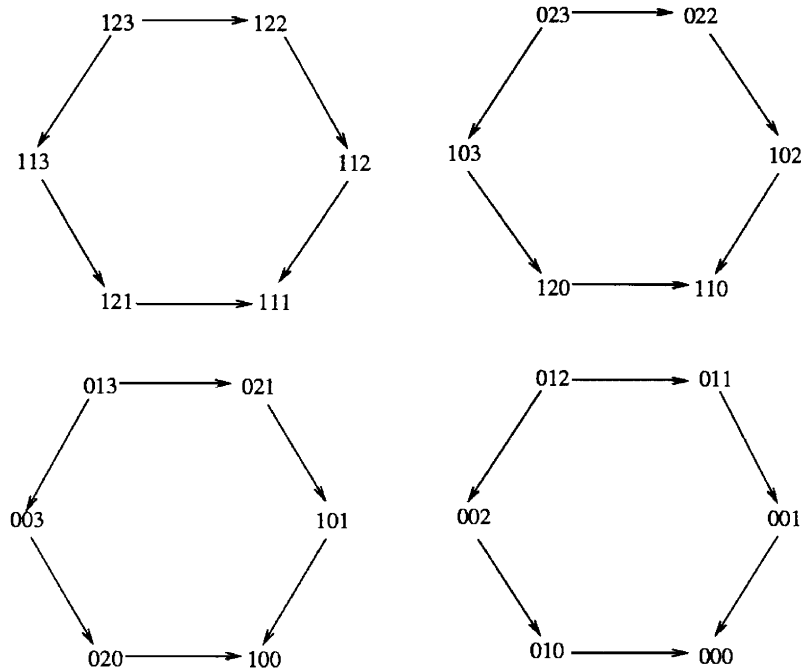
The graph  $G_{1,2}$  is depicted in Fig. 1. It may be seen that it has various connected components, all of them isomorphic and each one with exactly one out-vertex and exactly one in-vertex. This turns out to be true for all graphs  $G_{s,k}$ ,  $s, k \geq 1$ . To see that, the following property of  $G_{s,k}$  will be used.

**Lemma 4.3.** *Let  $(i_0, \dots, i_k)$  be an arbitrary out-vertex in the graph  $G_{s,k}$ . Then, any directed path in  $G_{s,k}$  from  $(i_0, \dots, i_k)$  to an in-vertex has length  $k(k+1)/2$ .*

**Proof.** Let us identify each entry  $i_p$  ( $p = 0, \dots, k$ ) with its initial position  $p$  in the  $(k+1)$ -tuple. As we move along a path in  $G_{s,k}$  that starts in this vertex, the  $p$ th-entry will change its value (to new values  $i'_p$ ) and the position it occupies in the  $(k+1)$ -tuple. The lemma follows from the fact that we will get an in-vertex when and only when, for any pair of entries  $p < q$ ,  $i'_p$  is to the right of  $i'_q$ . Indeed, suppose that, after several edges, there is a pair  $p < q$  such that the  $p$ th-entry  $i'_p$  is still to the left of the  $q$ th-entry  $i'_q$ . We then have  $i'_p \in \{i_p - p, \dots, i_p\}$ , with  $i'_p = i_p - p$  when all entries  $i_0, \dots, i_{p-1}$  have been moved to the right of  $i_p$ , and  $i'_p = i_p$  when none of these entries has been moved to the right of  $i_p$ . Suppose  $i'_p = i_p - t$  ( $t \in \{0, \dots, p\}$ ). In this case, we necessarily have  $i'_q \in \{i_q - (q-1-p+t), \dots, i_q\}$ , because  $q-1-p+t$  is the maximum number of positions that  $i_q$  can move to the left always keeping to the right of  $i'_p$ . It follows that

$$\begin{aligned} i'_q - i'_p &\geq i_q - (q - p - 1 + t) - i_p + t = i_q - i_p - q + p + 1 \\ &\geq q - p - q + p + 1 = 1 \end{aligned}$$

and, hence, the vertex is still not an in-vertex. On the other hand, it is clear that, when all such “transpositions” have been made, the resulting vertex is really an in-vertex.

Fig. 1. The graph  $G_{1,2}$ .

Now, there are  $k(k+1)/2$  such “transpositions” to be made. Since going through one directed edge in the graph corresponds to making exactly one of these “transpositions”, we conclude that we get an in-vertex after going over a directed path of length  $k(k+1)/2$  and only in this case.  $\square$

Using this lemma, we can prove the following result which will be used below to prove the coherence theorem, and which in particular shows that the connected components of  $G_{s,k}$  are parametrized by the injections  $\{0, 1, \dots, k\} \hookrightarrow \{0, 1, \dots, s+k\}$ , so that  $G_{s,k}$  has  $\binom{s+k+1}{k+1}$  connected components.

**Proposition 4.4.** *Let  $s, k \geq 1$ . Then, each connected component of  $G_{s,k}$  has exactly one out-vertex and one in-vertex. Furthermore, all its components are isomorphic and independent of  $s$ .*

**Proof.** Clearly, each component has at least one out-vertex (just follow an inversely directed path from any vertex in the component until the end). To prove that it has at most one, suppose there are two different out-vertices  $(i_0^{\text{out}}, \dots, i_k^{\text{out}})$  and  $(i_0'^{\text{out}}, \dots, i_k'^{\text{out}})$  in the same component  $C$ . In particular, they have different heights. Since there is no directed path connecting them (no directed path ends in an out-vertex), there must be

directed paths  $\gamma, \gamma'$  starting at each out-vertex which meet in some common vertex  $(i_0, \dots, i_k)$ . Following from this vertex a directed path  $\bar{\gamma}$  until the end, we will get an in-vertex  $(i_0^{\text{in}}, \dots, i_k^{\text{in}})$ . Now, by the previous proposition, all directed paths from an out- to an in-vertex have the same length, so that both composite paths  $\gamma\bar{\gamma}$  and  $\gamma'\bar{\gamma}$  have the same length. On the other hand, when going over any directed edge, the height always decreases by exactly one unit. It follows that the height of the final in-vertex should have two different values, which makes no sense. Hence, there is exactly one out-vertex in each component. It immediately follows then that there is also exactly one in-vertex in each component, with a well-defined value of its height, equal to the height of the corresponding out-vertex minus  $k(k+1)/2$ . To see that all connected components are isomorphic, let us denote by  $C(i_0^{\text{out}}, \dots, i_k^{\text{out}})$ ,  $C(i_0^{\text{out}'}, \dots, i_k^{\text{out}'})$  the connected components corresponding to the out-vertices  $(i_0^{\text{out}}, \dots, i_k^{\text{out}})$  and  $(i_0^{\text{out}'}, \dots, i_k^{\text{out}'})$ , respectively. Then, for any vertex  $(i_0, \dots, i_k)$  in  $C(i_0^{\text{out}}, \dots, i_k^{\text{out}})$ , we have  $(i_0, \dots, i_k) = \tau(i_0^{\text{out}}, \dots, i_k^{\text{out}})$ , for a suitable composite  $\tau$  of expanded coherers. Then, we get an isomorphism  $\varphi : C(i_0^{\text{out}}, \dots, i_k^{\text{out}}) \cong C(i_0^{\text{out}'}, \dots, i_k^{\text{out}'})$  by defining

$$\varphi(i_0^{\text{out}}, \dots, i_k^{\text{out}}) = (i_0^{\text{out}'}, \dots, i_k^{\text{out}'})$$

and for any other vertex

$$\varphi(\tau(i_0^{\text{out}}, \dots, i_k^{\text{out}})) = \tau'(i_0^{\text{out}'}, \dots, i_k^{\text{out}'}),$$

where  $\tau'$  is the composite of expanded coherers obtained from  $\tau$  by suitably changing the indices of the expanded coherers which appear in  $\tau$ , according to the corresponding initial out-vertex. Finally, to prove that the components are independent of  $s$ , it is enough to see, for ex., that the connected components  $C(1, \dots, k+1)$  of  $G_{s,k}$  and  $G_{s',k}$ , for any  $s, s' \geq 1$ , are isomorphic, and this follows immediately from the definition of both graphs.  $\square$

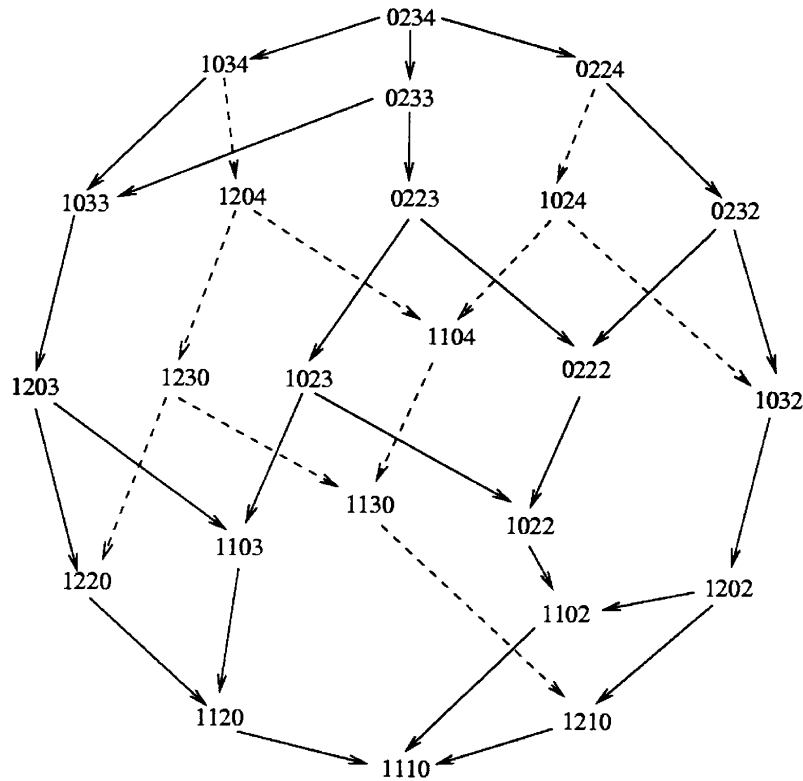
As example, it is shown in Fig. 2 the connected component  $C(0, 2, 3, 4)$  of the graph  $G_{1,3}$ , whose in-vertex is  $(1, 1, 1, 0)$ . Notice that it coincides with the 1-skeleton of the 3-dimensional permutohedron  $P_3$ , which we recall it is obtained from an octahedron by cutting out 6 small octahedra about its six vertices. Similarly, the connected components of  $G_{1,2}$  (cf. Fig. 1) were equal to the 1-skeleton of the 2-dimensional permutohedron  $P_2$ .<sup>4</sup> Using the coherence Theorem 4.2, it is shown below that this is always true (see Corollary 4.5).

Let us now prove the coherence theorem. We proceed in a way very similar that followed by MacLane to prove the classical coherence theorem for monoidal categories (see [9,10]).

**Proof of Theorem 4.2.** Let  $v = (i_k, \dots, i_0)$ ,  $v' = (i'_k, \dots, i'_0)$  be two arbitrary vertices in  $G_{s,k}$ , corresponding to two objects  $f, f'$  in  $\mathcal{C}_{s,k}$ . We have to see that any two different paths between them in  $G_{s,k}$  (if there exists any path at all) correspond to the same morphism in  $\mathcal{C}_{s,k}$ . We may assume that both vertices belong to the same connected

<sup>4</sup> I am very grateful to the referee for pointing out to me the permutohedron nature of the connected components of the graphs  $G_{s,k}$ .



Fig. 2. The connected component  $C(0, 2, 3, 4)$  of  $G_{1,3}$ .

component, because otherwise there is nothing to be shown. Let us denote by  $C_{s,k}$  this component and let  $v_{\text{in}} = (i_k^{\text{in}}, \dots, i_0^{\text{in}})$  be the corresponding in-vertex. We clearly have a directed path from each vertex  $v, v'$  to  $v_{\text{in}}$  that we may choose in a canonical way, say by always applying in each step the expanded coherer  $1_{g'} \circ \tau_{ij}^n \circ 1_g$  with the least possible value of  $n$  ( $n$  will be called the *laterality* of the expanded coherer). This, together with the fact that  $\mathcal{C}_{s,k}$  is a groupoid, reduces the proof of the theorem to see that any two directed paths from an arbitrary vertex  $v$  in  $C_{s,k}$  to the vertex  $v_{\text{in}}$  define the same isomorphism in  $\mathcal{C}_{s,k}$ . The proof is by induction on the height of  $v \neq v_{\text{in}}$ . Let  $h(v_{\text{in}}) = h_0$ . Hence,  $h(v) \in \{h_0 + 1, \dots, h_0 + \frac{1}{2}k(k+1)\}$ . If  $h(v) = h_0 + 1$ , there is only one path in  $C_{s,k}$  from  $v$  to  $v_{\text{in}}$  (a path of length one) and there is nothing to be shown. Suppose  $h(v) > h_0 + 1$ . We have to distinguish two cases, according to the rank of  $v$ . If  $r(v) = 1$ , there is again a unique directed edge starting at  $v$ . After crossing that edge, we get a new vertex  $v'$  whose height is  $h(v') = h(v) - 1$  and the result follows by the induction hypothesis. Suppose now that  $r(v) > 1$ . In this case, there are various directed edges starting at  $v$ , distinguished by the laterality of the corresponding expanded coherer. By the induction hypothesis, any two paths starting with the same

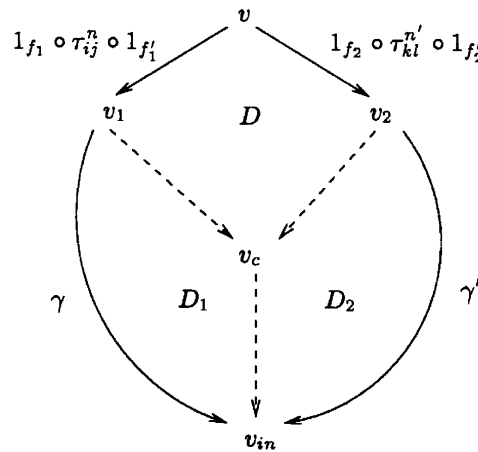


Fig. 3. Proof of Theorem 4.2.

directed edge in  $v$  will define the same morphism in  $\mathcal{C}_{s,k}$ , because this common first edge will decrease the height by a unit. Thus, it only remains to consider the case of two paths  $\gamma, \gamma'$  from  $v$  starting with different edges, of lateralities  $n$  and  $n'$ , with  $n \neq n'$ . The situation is depicted in Fig. 3.

It is clear from this figure that we just need to see that both initial edges can be made to converge to a common vertex  $v_c$  in such a way that the resulting diagram  $D$  commutes in  $\mathcal{C}_{s,k}$ , the corresponding bottom diagrams  $D_1, D_2$  being commutative by the induction hypothesis. There are two possibilities, according to the value of  $|n - n'|$ . If  $|n - n'| = 1$ , the convergence may be achieved through an hexagonal diagram, which commutes in  $\mathcal{C}_{s,k}$  by the coherence relations. If  $|n - n'| > 1$ , we need just to apply the expanded coherers with the lateralities interchanged to get a square which will be commutative in  $\mathcal{C}_{s,k}$  by the interchange law.  $\square$

Notice that, by the last paragraph in this proof, what it has actually been shown is that all closed paths in  $G_{s,k}$  are the boundary of a union of a certain number of instances of the hexagonal diagrams giving the coherence relations (hence, commutative) together with some quadrilaterals (commutative by the interchange law). Using this, the above-mentioned relation between our graphs  $G_{s,k}$  and the permutohedra easily follows. Let us first recall a few facts about the permutohedra, defined for the first time by Milgram [12] (see also [2], where they are called *zilchgons*). For any  $k \geq 1$ , the *permutohedron*  $P_k$  is defined as the convex hull of the set of points  $(\sigma(1), \dots, \sigma(k+1)) \in \mathbb{R}^{k+1}$  for all permutations  $\sigma \in S_{k+1}$ . It is shown [2] that  $P_k$  is a  $k$ -dimensional convex polyhedron whose  $(k+1-r)$ -dimensional faces, for all  $r = 1, \dots, k+1$ , are indexed by pairs  $(p, s)$ , where  $p$  is an ordered partition of  $\{1, \dots, k+1\}$ , i.e., a partition of the form

$$p = \{\{1, \dots, i_1\}, \{i_1 + 1, \dots, i_1 + i_2\}, \dots, \{i_1 + \dots + i_{r-1} + 1, \dots, i_1 + \dots + i_r\}\}$$

with  $i_1 + \dots + i_r = k + 1$  and all  $i_j \geq 1$ , and  $s$  is a shuffle of type  $(i_1, \dots, i_r)$ , namely, a permutation  $\sigma \in S_{k+1}$  such that  $\sigma(i) < \sigma(j)$  whenever  $i$  and  $j$  belong to the same block in the partition. This is equivalent to label the  $(k + 1 - r)$ -dimensional faces by ordered tuples  $(A_1, \dots, A_r)$  of nonempty disjoint subsets of  $\{1, 2, \dots, k + 1\}$  such that  $\bigcup_{i=1}^r A_i = \{1, \dots, k + 1\}$  (the tuple  $(A_1, \dots, A_r)$  corresponding to a pair  $(p, s)$  is obtained by applying the shuffle  $s$  to  $p$ ). In particular, it turns out that the 1-dimensional faces (case  $r = k$ ) are labelled by pairs  $(\sigma, \tau)$ , where  $\sigma$  is any permutation in  $S_{k+1}$ , and  $\tau$  is any transposition of the form  $\tau = (i, i + 1)$ , for some  $i \in \{1, \dots, k\}$ ;  $\tau$  gives, for the ordering defined by  $\sigma$ , the two point set in the corresponding tuple  $(A_1, \dots, A_k)$ . For example, if  $k = 3$ , the pairs  $((123), (34))$ ,  $((14), (12))$ , respectively, correspond to the tuples  $(\{2\}, \{3\}, \{1, 4\})$  and  $(\{4, 2\}, \{3\}, \{1\})$ . Such a pair  $(\sigma, (i, i + 1))$  represents an edge in  $P_k$  between the vertices  $(\sigma(1), \dots, \sigma(i), \sigma(i + 1), \dots, \sigma(k + 1))$  and  $(\sigma(1), \dots, \sigma(i + 1), \sigma(i), \dots, \sigma(k + 1))$ . It follows that the 1-skeleton of  $P_k$  is nothing but the Cayley graph  $\text{Cay}(S_{k+1})$  of  $S_{k+1}$  with respect to the generators  $\{(12), (23), \dots, (k, k + 1)\}$  (for the definition of the Cayley graph of a group, see for e.g. [3]). We then have the following result, which suggests the name *cosemisimplihedra* for the permutohedra:

**Corollary 4.5.** *For any  $k \geq 1$ , the connected components of  $G_{s,k}$  are isomorphic to the 1-skeleton of  $P_k$ .*

**Proof.** It is enough to see that the connected component  $C(1, 2, \dots, k + 1)$  of  $G_{1,k}$ , for example, is isomorphic to  $\text{Cay}(S_{k+1})$ . If  $C(1, \dots, k + 1)_0$  and  $\text{Cay}(S_{k+1})_0 = S_{k+1}$  denote the respective sets of vertices in both graphs, let us define a map  $\Phi : C(1, \dots, k + 1)_0 \rightarrow \text{Cay}(S_{k+1})_0$  by

$$\Phi(i_0, \dots, i_k) = (n_1, n_1 + 1)(n_2, n_2 + 1) \cdots (n_r, n_r + 1),$$

where  $n_1, \dots, n_r$  are the lateralities of the successive expanded coherers needed to go from the out-vertex  $(1, 2, \dots, k + 1)$  to  $(i_0, \dots, i_k)$ . Although there are can be several paths in  $G_{s,k}$  from one vertex to the other, the corresponding permutation is uniquely defined. Indeed, according to the remark after the proof of the coherence theorem, any two such paths are joined through some hexagonal and/or quadrilateral faces. Now, the hexagonal faces just correspond to the relation

$$(i, i + 1)(i + 1, i + 2)(i, i + 1) = (i + 1, i + 2)(i, i + 1)(i + 1, i + 2),$$

$$i = 1, \dots, k - 1$$

in the symmetric group, while the quadrilaterals correspond to the relation

$$(i, i + 1)(j, j + 1) = (j, j + 1)(i, i + 1), \quad |i - j| \geq 2.$$

Furthermore, this map is injective, because if two vertices in  $C(1, 2, \dots, k + 1)_0$  are mapped to the same permutation, the two formally different decompositions of the permutation must be related through the previous relations. But this means that the corresponding paths in  $C(1, \dots, k + 1)$  must be related by hexagonal and quadrilateral faces as before, so that they necessarily define a closed path, both final vertices being equal. Since both sets of vertices have the same cardinal, it follows that it is a bijection, and it clearly preserves the edges.  $\square$

To finish this section, it is worth emphasizing that, contrary to what it might seem at first sight, our definition of 2-cosemisimplicial object in  $\mathfrak{C}$  is not completely equivalent to a pseudofunctor  $\mathcal{F} : \Delta_s \rightarrow \mathfrak{C}$ , where  $\Delta_s$  is the semisimplicial category viewed as a 2-category with only the identity 2-morphisms.<sup>5</sup> It is known that a pseudofunctor  $\mathcal{F} : \mathfrak{C} \rightarrow \mathfrak{D}$ , with  $\mathfrak{C}$  and  $\mathfrak{D}$  2-categories, is equivalent to a 2-functor  $F : H(\mathfrak{C}) \rightarrow \mathfrak{D}$ , where  $H(\mathfrak{C})$  is a suitable 2-category which depends on  $\mathfrak{C}$  but not on  $\mathcal{F}$ . More precisely, it turns out that the inclusion functor  $\mathbf{2-Cat} \hookrightarrow \mathbf{2-Cat}_{ps}$ , where  $\mathbf{2-Cat}_{ps}$  and  $\mathbf{2-Cat}$  are the categories with objects all (small) 2-categories and morphisms all pseudofunctors or all 2-functors, respectively, has a left adjoint  $H : \mathbf{2-Cat}_{ps} \rightarrow \mathbf{2-Cat}$  (cf. [8, Proposition 4.2]).<sup>6</sup> The 2-category  $H(\mathfrak{C})$  is in some sense obtained from  $\mathfrak{C}$  by making it free with respect to 1-morphisms. Explicitly, it has as objects the same as  $\mathfrak{C}$ , as 1-morphisms all finite sequences of composable 1-morphisms in  $\mathfrak{C}$  (included the empty sequence if both the domain and codomain objects coincide) and as 2-morphisms between two such paths all 2-morphisms in  $\mathfrak{C}$  between the composite 1-morphisms defined by each path (the composite 1-morphism being the identity when the path is the empty sequence). Composition of 1-morphisms is given by concatenation and compositions of 2-morphisms by those in  $\mathfrak{C}$  in the obvious way. If  $\mathfrak{C} = \Delta_s$ , the corresponding  $H(\Delta_s)$ , which will be denoted by  $2\Delta_s$ , is a 2-category where, for any two 1-morphisms, we still have at most one 2-morphism between them (actually, a 2-isomorphism), but they are no longer identities all of them. For example, for any pair  $i, j$  such that  $0 \leq i < j \leq n+1$  and any  $n \geq 1$ , there is in  $2\Delta_s$  a (nonidentity) 2-isomorphism  $\beta_{ij}^n : (e_n^i, e_{n+1}^j) \Rightarrow (e_n^{j-1}, e_{n+1}^i)$ , where the  $e_n^i : [n-1] \hookrightarrow [n]$  denote the usual face morphisms in  $\Delta_s$ . In  $2\Delta_s$ , we have a sub-2-category  $2\Delta_s^0$  with the same objects as  $2\Delta_s$ , with  $2\Delta_s^0([n-1], [n]) = 2\Delta_s([n-1], [n])$  for all  $n \geq 1$ , but with  $2\Delta_s^0([n-1], [n+k])$ , for all  $n, k \geq 1$ , equal to the full subcategory of  $2\Delta_s([n-1], [n+k])$  whose objects are only the sequences of composable face morphisms, i.e., sequences of length  $k+1$  of the form  $(e_n^{i_0}, \dots, e_{n+k}^{i_k})$ . Notice that this sub-2-category is biequivalent to  $2\Delta_s$ , but not equal. Thus, in  $2\Delta_s^0$  the only 1-morphisms defined by sequences of length 1 are those given by a face morphism in  $\Delta_s$ , while in  $2\Delta_s$  we further have all sequences of the form  $(f)$ , for  $f$  any composite of more than one face morphism. We have then the following:

**Proposition 4.6.** *For any 2-category  $\mathfrak{C}$ , a 2-cosemisimplicial object in  $\mathfrak{C}$  as defined in Definition 4.1 is equivalent to a 2-functor  $F : 2\Delta_s^0 \rightarrow \mathfrak{C}$ .*

**Proof.** For any 2-category  $\mathfrak{D}$ , a 2-functor  $F : \mathfrak{D} \rightarrow \mathfrak{C}$  is completely defined by the images of the 2-morphisms in any pair of families of the following type: (1) a family  $\mathcal{A}_{\mathfrak{D}}$  of “generating” 2-morphisms in  $\mathfrak{D}$ , by which we mean nonidentity 2-morphisms  $\{\tau_{\lambda}\}_{\lambda}$  in  $\mathfrak{D}$  such that any nonidentity 2-morphism in  $\mathfrak{D}$  can be obtained as a (non-necessarily unique) pasting of the  $\tau_{\lambda}$  and identity 2-morphisms, and (2) a family  $\mathcal{B}_{\mathfrak{D}}$  of identity 2-morphisms  $\{1_{f_{\mu}}\}_{\mu}$  such that any 1-morphism in  $\mathfrak{D}$  can be obtained as a

<sup>5</sup> This observation has been motivated by a comment of the referee.

<sup>6</sup> I acknowledge the referee for calling my attention to this result, which seems to be well known among 2-category specialists.

(nonnecessarily unique) composition of the  $f_\mu$ . This is because a 2-functor preserves vertical and horizontal compositions (hence, also pastings) and the identity 2-morphisms, together with the fact that  $1_{g \circ f} = 1_g \circ 1_f$ . Furthermore, the images of these 2-morphisms can be chosen arbitrarily except that all possible relations between them have to be preserved. If  $\mathfrak{D} = 2\mathcal{A}_s^0$ , a pair of families as above is given by

$$\mathcal{A}_{2\mathcal{A}_s^0} = \{\beta_{ij}^n, \quad 0 \leq i < j \leq n+1, \quad n \geq 1\},$$

$$\mathcal{B}_{2\mathcal{A}_s^0} = \{1_{\varepsilon_n^i}, \quad i = 0, 1, \dots, n, \quad n \geq 1\}.$$

Indeed, the face morphisms generate all morphisms in  $\mathcal{A}_s$  and, given two sequences  $(\varepsilon_n^{i_0}, \dots, \varepsilon_{n+k}^{i_k}), (\varepsilon_n^{i'_0}, \dots, \varepsilon_{n+k}^{i'_k})$  defining the same composite morphism in  $\mathcal{A}_s$ , they can be connected by a pasting of the  $\beta_{ij}^r$  to the common canonical decomposition  $(\varepsilon_n^{j_0}, \dots, \varepsilon_{n+k}^{j_k})$  with  $j_0 < j_1 < \dots < j_k$ . Hence, the corresponding (unique) 2-morphism between both sequences is really a pasting of the  $\beta_{ij}^r$ . It also follows from the uniqueness of the 2-morphisms in  $2\mathcal{A}_s^0$  that the  $\beta_{ij}^n$  satisfy the relations

$$\begin{aligned} & (1_{\varepsilon_{n+2}^i} \circ \beta_{j-1, k-1}^n) \cdot (\beta_{ik}^{n+1} \circ 1_{\varepsilon_n^{j-1}}) \cdot (1_{\varepsilon_{n+2}^k} \circ \beta_{ij}^n) \\ &= (\beta_{ij}^{n+1} \circ 1_{\varepsilon_{n+2}^{k-2}}) \cdot (1_{\varepsilon_{n+2}^j} \circ \beta_{i, k-1}^n) \cdot (\beta_{jk}^{n+1} \circ 1_{\varepsilon_n^i}) \end{aligned}$$

for all  $0 \leq i < j < k \leq n+2$  and all  $n \geq 1$ . Moreover, our coherence theorem (Theorem 4.2) implies that any other relation between the  $\beta_{ij}^n$  is a consequence of these relations. Hence, giving a 2-functor  $F : 2\mathcal{A}_s^0 \rightarrow \mathfrak{C}$  is indeed equivalent to give arbitrary 1-morphisms  $\partial_n^i$  in  $\mathfrak{C}$  (the images of the 2-morphisms  $1_{\varepsilon_n^i}$ ) and 2-morphisms  $\tau_{ij}^n$  (the images of the 2-morphisms  $\beta_{ij}^n$ ) in  $\mathfrak{C}$  satisfying the coherence relations in Definition 4.1.  $\square$

Such a 2-functor  $F : 2\mathcal{A}_s^0 \rightarrow \mathfrak{C}$ , however, will not extend uniquely to a 2-functor  $\tilde{F} : 2\mathcal{A}_s \rightarrow \mathfrak{C}$ . Thus,  $2\mathcal{A}_s$  contains 2-morphisms  $(f) \Rightarrow (\varepsilon_n^{i_0}, \dots, \varepsilon_{n+k}^{i_k})$  with  $f = \varepsilon_{n+k}^{i_k} \circ \dots \circ \varepsilon_n^{i_0}$ ,  $k \geq 1$  which are not given by a pasting of the  $\beta_{ij}^n$  and, hence, such that their images are not determined by the images of the  $\beta_{ij}^n$ . The reader may easily check that a right family  $\mathcal{A}_{2\mathcal{A}_s}$  of generating 2-morphisms for  $2\mathcal{A}_s$  is given by the family  $\mathcal{A}_{2\mathcal{A}_s^0}$  together with the 2-morphisms

$$\alpha_{i_0, \dots, i_k}^n : (\varepsilon_{n+k}^{i_k} \circ \dots \circ \varepsilon_n^{i_0}) \Rightarrow (\varepsilon_n^{i_0}, \dots, \varepsilon_{n+k}^{i_k})$$

for all  $0 \leq i_0 < i_1 < \dots < i_k \leq n+k$  and all  $n \geq 1$ . To define a 2-functor  $\tilde{F} : 2\mathcal{A}_s \rightarrow \mathfrak{C}$ , and hence a pseudofunctor  $\mathcal{F} : \mathcal{A}_s \rightarrow \mathfrak{C}$ , we will need to give images of the 2-morphisms in this additional family satisfying the appropriate relations. It seems possible, however, that all extensions  $\tilde{F} : 2\mathcal{A}_s \rightarrow \mathfrak{C}$  of  $F$  turn out to be equivalent in a suitable sense (in the same way as the extension of a functor defined on the skeleton of a category to the whole category is unique up to isomorphism), but we did not explore that any further.

## 5. Cochain complexes from 2-cosemisimplicial objects in $\text{Cat}_K$

Given a cosemisimplicial object in an abelian category, it is usual to consider the corresponding cochain complex and cohomology. Hence, the following question

naturally raises: what are the analogs of these cochain complexes and their cohomologies in the case of a 2-cosemisimplicial object in a 2-category? As in the categorical setting, it is expected that finding these analogs will require restricting to suitable *abelian 2-categories*, for which hypothetical *2-cochain complexes* will make sense. However, we will not pursue this direction here. Instead, the purpose of this section is to show that usual cochain complexes of  $K$ -modules may still be constructed from certain *enhanced 2-cosemisimplicial objects* in a particular 2-category. Namely, the 2-category  $\mathbf{Cat}_K$  having as objects the (small)  $K$ -linear categories, as 1-morphisms the  $K$ -linear functors and as 2-morphisms the natural transformations. As an example, which was our original motivation, we show in the next section that the purely pseudofunctorial deformation complex introduced in [6] for any  $K$ -linear pseudofunctor  $\mathcal{F}$  may be obtained in this way from a suitable enhanced 2-cosemisimplicial object in  $\mathbf{Cat}_K$  associated to  $\mathcal{F}$ .

Suppose we are given a 2-cosemisimplicial object  $\mathcal{C}^\bullet$  in  $\mathbf{Cat}_K$  and let  $F_n^i : \mathcal{C}^{n-1} \rightarrow \mathcal{C}^n$  ( $i = 0, 1, \dots, n, n \geq 1$ ) and  $\tau_{ij}^n : F_{n+1}^j \circ F_n^i \Rightarrow F_{n+1}^i \circ F_n^{j-1}$  ( $0 \leq i < j \leq n+1, n \geq 1$ ) be the corresponding coface functors and cosemisimplicial coherers, which are natural isomorphisms in this case. To simplify notation, we shall write  $F_n^{i_0, \dots, i_k}$  to denote the composite functor  $F_{n+k}^{i_k} \circ F_{n+k-1}^{i_{k-1}} \circ \dots \circ F_n^{i_0}$  ( $n, k \geq 1$ ). According to Theorem 4.2, for all  $n, k \geq 1$  and  $(i_0, \dots, i_k) \neq (j_0, \dots, j_k)$ , with  $i_q, j_q \in \{0, 1, \dots, n+q\}$  and  $q = 0, 1, \dots, k$ , there exists at most one canonical natural isomorphism from  $F_n^{i_0, \dots, i_k}$  to  $F_n^{j_0, \dots, j_k}$ , given by pasting the appropriate coherers  $\tau_{ij}^n$ 's and/or its inverses. It will be denoted by  $\tau_{(i_0, \dots, i_k), (j_0, \dots, j_k)}^n$ . Notice that such canonical isomorphisms may not exist, depending on the  $(k+1)$ -tuples  $(i_0, \dots, i_k)$  and  $(j_0, \dots, j_k)$ . This is because, as seen before, the graph  $G_{n,k}$  is not connected. For example, there is no canonical path between  $F_n^{1,1}$  and  $F_n^{0,0}$  nor between  $F_n^{1,1,0,3}$  and  $F_n^{1,2,3,4}$ . When  $(i_0, \dots, i_k) = (j_0, \dots, j_k)$ , we will agree that  $\tau_{(i_0, \dots, i_k), (i_0, \dots, i_k)}^n$  denotes the corresponding identity natural transformation.

Roughly, the method of getting cochain complexes of  $K$ -modules from the 2-cosemisimplicial object  $\mathcal{C}^\bullet$  consists of the following. For all  $n \geq 0$ , choose a pair of objects  $X_n, X'_n$  in each category  $\mathcal{C}^n$ , take for each such pair the corresponding  $K$ -modules of morphisms  $\text{Hom}_{\mathcal{C}^n}(X_n, X'_n)$  (they are indeed  $K$ -modules because  $\mathcal{C}^n$  is  $K$ -linear) and define coboundary maps between them using the coface functors  $F_n^i$ , which are  $K$ -linear. More explicitly, we would like these coboundary maps  $\delta : \text{Hom}_{\mathcal{C}^{n-1}}(X_{n-1}, X'_{n-1}) \rightarrow \text{Hom}_{\mathcal{C}^n}(X_n, X'_n)$  to be of the form

$$\delta(\varphi) \approx \sum_{i=0}^n (-1)^i F_n^i(\varphi) \quad (5.1)$$

for all  $\varphi \in \text{Hom}_{\mathcal{C}^{n-1}}(X_{n-1}, X'_{n-1})$ . This procedure, however, makes no sense in general, because the  $F_n^i(\varphi)$  belong to different  $K$ -modules of morphisms for different values of  $i \in \{0, 1, \dots, n\}$  (they have different domains and codomains). This could be easily overcome if all such domains and codomains were (canonically) isomorphic to the corresponding reference objects  $X_n$  and  $X'_n$ , respectively, because we can then get morphisms in  $\text{Hom}_{\mathcal{C}^n}(X_n, X'_n)$  by just taking the composite of each term  $F_n^i(\varphi)$  with the appropriate (canonical) isomorphisms on the left and on the right. However, this will not be true for randomly chosen objects  $X_n$  and  $X'_n$ . One may try to fix that by

choosing an object  $X \in |\mathcal{C}^0|$  and taking  $X_n$  and  $X'_n$ , for all  $n \geq 1$ , equal to some iterated images of  $X$  by the coface functors  $F_n^i$ . For example, for  $n \geq 1$ , we could inductively define

$$X_n := F_n^n(X_{n-1}), \quad (5.2)$$

$$X'_n := F_n^{n-1}(X'_{n-1}) \quad (5.3)$$

with  $X_0 = X'_0 = X$ . In this way, both the domain and codomain of  $F_n^i(\varphi)$ , for all  $i = 0, \dots, n$ , will be of the form  $F_1^{i_0, \dots, i_{n-1}}$  for some  $n$ -tuples of positive integers  $(i_0, \dots, i_{n-1})$ , so that they can be related via the natural isomorphisms  $\tau_{ij}^n$ . Even in this way, however, the problem turns out to persist because of the non-connectedness of the graphs  $G_{1,n-1}$ . Actually, the problem persists independently of how the references  $X_n$  and  $X'_n$  are chosen among all possible iterated images of  $X$ . This is easily seen by considering the cases  $n = 1$  and  $2$ . Suppose we take  $X_1 = F_1^1(X)$ . Then, for any  $\varphi : X_1 \rightarrow X'_1$ , the domains of  $F_2^0(\varphi)$ ,  $F_2^1(\varphi)$  and  $F_2^2(\varphi)$  will, respectively, be  $F_1^{1,0}(X)$ ,  $F_1^{1,1}(X)$  and  $F_1^{1,2}(X)$ . But a glance to the graph  $G_{1,1}$  immediately shows that there is no choice for  $X_2 = F_2^i(X_1)$  such that it is simultaneously canonically isomorphic to these three domains.

The above discussion shows that to define cochain complexes by this method, with the coboundary maps given by Eq. (5.1), we need some additional hypothesis on the 2-cosemisimplicial object  $\mathcal{C}^\bullet$ . This leads us to introduce the following definition.

**Definition 5.1.** Let  $\mathcal{C}$  be any 2-category. By an enhanced 2-cosemisimplicial object in  $\mathcal{C}$  we shall mean a 2-cosemisimplicial object  $(X^\bullet, \partial, \tau)$  in  $\mathcal{C}$  together with a 2-isomorphism  $\phi : \partial_1^1 \Rightarrow \partial_1^0$  such that

$$\tau_{0,1}^1 \cdot (1_{\partial_2^1} \circ \phi) \cdot \tau_{1,2}^1 = (1_{\partial_2^0} \circ \phi) \cdot \tau_{0,2}^1 \cdot (1_{\partial_2^0} \circ \phi). \quad (5.4)$$

As the coherence relations on the  $\tau_{ij}^n$ 's, the above condition on  $\phi$  is related to a coherence theorem. To state this theorem, let us denote by  $\mathcal{C}_{1,k}^\phi$ , for all  $k \geq 1$ , the subcategory of  $\mathcal{C}(X^0, X^{k+1})$  with objects the same as in  $\mathcal{C}_{1,k}$  (namely, the  $\partial$ -paths), but whose morphisms are all possible composites of expanded coherers of  $X^\bullet$  and expansions of  $\phi$  (i.e., 2-isomorphisms of the form  $1_f \circ \phi : f \circ \partial_1^1 \Rightarrow f \circ \partial_1^0$  for some  $\partial$ -path  $f$ ). The new coherence theorem states then the following:

**Theorem 5.2.** Let  $k \geq 1$ . Then, for any two objects  $f, f'$  in  $\mathcal{C}_{1,k}^\phi$ , there is one and only one morphism (actually an isomorphism) in  $\mathcal{C}_{1,k}^\phi$  from  $f$  to  $f'$ .

**Proof.** Let  $G_{1,k}^\phi$  be the graph with vertices all  $\partial$ -paths  $f : X^0 \rightarrow X^{k+1}$  and edges all the expanded coherers and expansions of  $\phi$ . In particular,  $G_{1,k}^\phi$  contains  $G_{1,k}$  as a subgraph (see Fig. 4 for the case  $k = 2$ ). As in the previous section, we identify a  $\partial$ -path  $f$  with the corresponding  $(k+1)$ -tuple  $(i_0, \dots, i_k)$ . Let us first prove that, given two arbitrary vertices  $(i_0, \dots, i_k)$  and  $(i'_0, \dots, i'_k)$ , there always exist a path in  $G_{1,k}^\phi$  between them. Clearly, it is enough to prove the assertion in the special case when

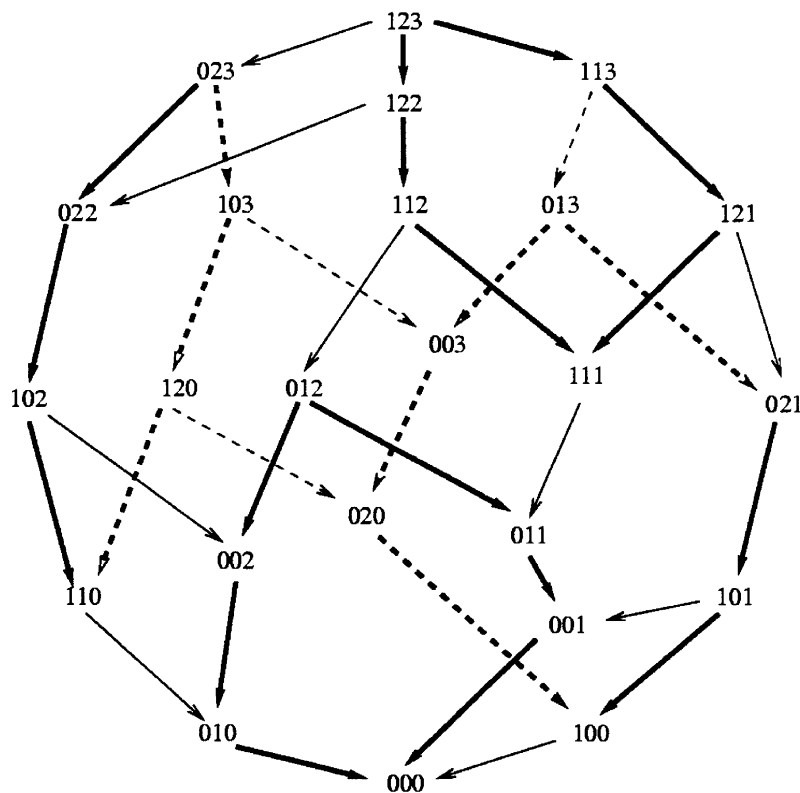


Fig. 4. The graph  $G_{1,2}^\phi$  (the edges in the four hexagons defining  $G_{1,2}$  are drawn in bold solid or dashed arrows to distinguish them from the additional edges corresponding to the expansions of  $\phi$ ).

both vertices are in-vertices. Otherwise, one immediately obtains the desired path by connecting each vertex to the corresponding in-vertex and adding any path between both in-vertices. To prove the claim for two in-vertices, observe that all in-vertices in  $G_{1,k}^\phi$  are of the form  $(1, \dots, 1, 0, \dots, 0)$ , the number of 1's plus the number of 0's being equal to  $k + 1$ . Starting at any such in-vertex and via the appropriate expansion of  $\phi$ , we can move to the neighbor “dual” vertex  $(0, 1, \dots, 1, 0, \dots, 0)$ , differing from it just in the first component. This is not an in-vertex, but it can be connected to the corresponding in-vertex through a path of expanded coherers. This new in-vertex will have one more zero than the initial one. Iterating this process, one finally gets the in-vertex  $(0, \dots, 0)$ . Since this may be done for any initial in-vertex, we conclude that two arbitrary in-vertices are indeed connected in  $G_{1,k}^\phi$ . Let us now prove that all paths in  $G_{1,k}^\phi$  between two arbitrary vertices  $(i_0, \dots, i_k)$  and  $(i'_0, \dots, i'_k)$  define the same morphism in  $\mathcal{C}_{1,k}^\phi$ . Since  $\mathcal{C}_{1,k}^\phi$  is also a groupoid, it suffices to prove the assertion when  $(i'_0, \dots, i'_k) = (0, \dots, 0)$ . We proceed again by induction on the height of  $v = (i_0, \dots, i_k)$ .



If  $h(v) = 0$ ,  $v$  necessarily coincides with  $(0, \dots, 0)$  and there is nothing to be shown. Suppose then that  $h(v) \geq 1$  and let  $\gamma, \gamma'$  be two directed paths starting at  $v$ . If the first edges in both  $\gamma$  and  $\gamma'$  coincide, the result follows directly by induction. Otherwise, the argument is similar to that used in the proof of Theorem 4.2. Namely, we see that both initial edges can be made to converge to a common vertex  $v_c$  in such a way that the resulting diagram  $D$  commutes in  $\mathcal{C}_{1,k}^\phi$ . We have to distinguish three possibilities:

(i) Both first edges are two different expanded coherers: in this case, the convergence is achieved in exactly the same way as in the proof of Theorem 4.2.

(i) One edge is an expansion  $1_f \circ \phi$  of  $\phi$  while the other one is an expanded coherer  $1_{f'} \circ \tau_{ij}^s \circ 1_{f''}$  with laterality  $s \geq 2$ : in this case the commutative diagram  $D$  is the square defined by the equality

$$(1_{f'} \circ \tau_{ij}^s \circ 1_{f''}) \cdot (1_f \circ \phi) = (1_{\tilde{f}} \circ \phi) \cdot (1_{f'} \circ \tau_{ij}^s \circ 1_{f''})$$

which holds by the interchange law.

(i) One edge is an expansion  $1_f \circ \phi$  of  $\phi$  while the other one is an expanded coherer of the form  $1_{f'} \circ \tau_{ij}^1$ , with laterality  $s = 1$ : in this case the commutative diagram  $D$  is just that defined by Eq. (5.4), which holds by hypothesis.  $\square$

These unique isomorphisms between the objects in  $\mathcal{C}_{1,k}^\phi$  will be called the *canonical enhanced 2-isomorphisms* and denoted by  $\tau_{(i_0, \dots, i_k), (j_0, \dots, j_k)}^\phi$ . Notice that, when the pair  $(i_0, \dots, i_k), (j_0, \dots, j_k)$  is such that there already exists a path of expanded coherers in  $G_{1,k}$  between the corresponding  $\partial$ -paths, this canonical enhanced 2-isomorphism  $\tau_{(i_0, \dots, i_k), (j_0, \dots, j_k)}^\phi$  coincides with the canonical 2-isomorphism  $\tau_{(i_0, \dots, i_k), (j_0, \dots, j_k)}^1$  defined in the previous section.

**Remark 5.3.** Surprisingly, the graph  $G_{1,2}^\phi$  turns out to be isomorphic to the connected components of  $G_{1,3}$  and, hence, to the 1-skeleton of the 3-dimensional permutohedron  $P_3$  (cf. Figs. 2 and 4). This suggests that the same may be true for all  $k \geq 2$ .

We may now carry out the above program. Let  $(\mathcal{C}^\bullet, F, \tau, \phi)$  be an enhanced 2-cosemisimplicial object in  $\mathbf{Cat}_K$  and let us fix an object  $X \in |\mathcal{C}^0|$ . For all  $n \geq 1$ , choose once and for all  $n$ -tuples of nonnegative integers  $(\mu_1^n, \dots, \mu_n^n)$  and  $(\nu_1^n, \dots, \nu_n^n)$ , with  $\mu_q^n, \nu_q^n \in \{0, \dots, q\}$ , and define objects  $X_n, X'_n \in |\mathcal{C}^n|$  by

$$X_n = F_1^{\mu_1^n, \dots, \mu_n^n}(X),$$

$$X'_n = F_1^{\nu_1^n, \dots, \nu_n^n}(X).$$

They will be called the *domain* and *codomain reference objects* in  $\mathcal{C}^n$ , respectively. According to the previous theorem, for all  $n \geq 1$  and  $i = 0, 1, \dots, n$ , we have canonical enhanced 2-isomorphisms

$$\tau_{(\nu_1^{n-1}, \dots, \nu_{n-1}^{n-1}, i), (\nu_1^n, \dots, \nu_{n-1}^n, \nu_n^n)}^\phi : F_1^{\nu_1^{n-1}, \dots, \nu_{n-1}^{n-1}, i} \Rightarrow F_1^{\nu_1^n, \dots, \nu_{n-1}^n, \nu_n^n},$$

$$\tau_{(\mu_1^n, \dots, \mu_{n-1}^n, \mu_n^n), (\mu_1^{n-1}, \dots, \mu_{n-1}^{n-1}, i)}^\phi : F_1^{\mu_1^n, \dots, \mu_{n-1}^n, \mu_n^n} \Rightarrow F_1^{\mu_1^{n-1}, \dots, \mu_{n-1}^{n-1}, i}.$$

Hence, by taking the corresponding  $X$ -components, we get isomorphisms

$$\alpha_{i,n}^\phi \equiv \left( \tau_{(v_1^{n-1}, \dots, v_{n-1}^{n-1}, i), (v_1^n, \dots, v_{n-1}^n, v_n^n)}^\phi \right)_X : F_n^i(X'_{n-1}) \rightarrow X'_n,$$

$$\beta_{i,n}^\phi \equiv \left( \tau_{(\mu_1^n, \dots, \mu_{n-1}^n, \mu_n^n), (\mu_1^{n-1}, \dots, \mu_{n-1}^{n-1}, i)}^\phi \right)_X : X_n \rightarrow F_n^i(X_{n-1}).$$

Let us further denote by  $M^n$ , for all  $n \geq 1$ , the  $K$ -module of morphisms

$$M^n := \text{Hom}_{\mathcal{C}^n}(X_n, X'_n).$$

We have then the following:

**Theorem 5.4.** *The above  $K$ -modules  $M^1, M^2, \dots$  together with the coboundary maps  $\delta : M^{n-1} \rightarrow M^n$ ,  $n \geq 2$ , given by*

$$\delta(\varphi) = \sum_{i=0}^n (-1)^i \alpha_{i,n}^\phi \circ F_n^i(\varphi) \circ \beta_{i,n}^\phi \quad (5.5)$$

*define a cochain complex. Furthermore, the cochain complexes defined in this way by different choices of the reference objects  $X_n, X'_n$ ,  $n \geq 1$ , and by different objects  $Y \in |\mathcal{C}^0|$  isomorphic to  $X$  are all isomorphic.*

**Proof.** Since the functors  $F_n^i$  are  $K$ -linear, the coboundary maps are indeed  $K$ -linear. To see that  $\delta^2 = 0$ , notice first that, by the naturality of the  $\tau_{ij}^n$ 's, we have

$$(F_{n+1}^j \circ F_n^i)(\varphi) = (\tau_{ij}^n)^{-1}_{X'_{n-1}} \circ (F_{n+1}^i \circ F_n^{j-1})(\varphi) \circ (\tau_{ij}^n)_{X_{n-1}}$$

for any  $\varphi : X_{n-1} \rightarrow X'_{n-1}$  and all  $0 \leq i < j \leq n+1$ . Then, proceeding in the usual way, we have

$$\begin{aligned} \delta^2(\varphi) &= \sum_{0 \leq i < j \leq n+1} (-1)^{i+j} \alpha_{j,n+1}^\phi \circ F_{n+1}^j(\alpha_{i,n}^\phi) \circ (\tau_{ij}^n)^{-1}_{X'_{n-1}} \\ &\quad \circ (F_{n+1}^i \circ F_n^{j-1})(\varphi) \circ (\tau_{ij}^n)_{X_{n-1}} \circ F_{n+1}^j(\beta_{i,n}^\phi) \circ \beta_{j,n+1}^\phi \\ &\quad + \sum_{n \geq i \geq j \geq 0} (-1)^{i+j} \alpha_{j,n+1}^\phi \circ F_{n+1}^j(\alpha_{i,n}^\phi) \circ (F_{n+1}^j \circ F_n^i)(\varphi) \circ F_{n+1}^j(\beta_{i,n}^\phi) \circ \beta_{j,n+1}^\phi \\ &= \sum_{0 \leq j \leq i \leq n} (-1)^{i+j+1} \alpha_{i+1,n+1}^\phi \circ F_{n+1}^{i+1}(\alpha_{j,n}^\phi) \circ (\tau_{j,i+1}^n)^{-1}_{X'_{n-1}} \\ &\quad \circ (F_{n+1}^j \circ F_n^i)(\varphi) \circ (\tau_{j,i+1}^n)_{X_{n-1}} \circ F_{n+1}^{i+1}(\beta_{j,n}^\phi) \circ \beta_{i+1,n+1}^\phi \\ &\quad + \sum_{n \geq i \geq j \geq 0} (-1)^{i+j} \alpha_{j,n+1}^\phi \circ F_{n+1}^j(\alpha_{i,n}^\phi) \circ (F_{n+1}^j \circ F_n^i)(\varphi) \circ F_{n+1}^j(\beta_{i,n}^\phi) \circ \beta_{j,n+1}^\phi, \end{aligned}$$

the last equality being obtained by a suitable reindexation in the first sum. Hence, the proof reduces to see that the  $\alpha$ 's,  $\beta$ 's and  $\tau$ 's satisfy the equations

$$\alpha_{j,n+1}^\phi \circ F_{n+1}^j(\alpha_{i,n}^\phi) \circ (\tau_{j,i+1}^n)_{X'_{n-1}} = \alpha_{i+1,n+1}^\phi \circ F_{n+1}^{i+1}(\alpha_{j,n}^\phi), \quad (5.6)$$

$$(\tau_{j,i+1}^n)_{X_{n-1}} \circ F_{n+1}^{i+1}(\beta_{j,n}^\phi) \circ \beta_{i+1,n+1}^\phi = F_{n+1}^j(\beta_{i,n}^\phi) \circ \beta_{j,n+1}^\phi \quad (5.7)$$

for all  $0 \leq j \leq i \leq n$  ( $n \geq 2$ ). Now, from the very definition of all the involved terms, we have that the left-hand side in the first equality is nothing but the  $X$ -component of the canonical enhanced 2-isomorphism

$$\tau_{(v_1^n, \dots, v_n^n, j), (v_1^{n+1}, \dots, v_{n+1}^{n+1})}^\phi \cdot \left( 1_{F_{n+1}^j} \circ \tau_{(v_1^{n-1}, \dots, v_{n-1}^{n-1}, i), (v_1^n, \dots, v_{n-1}^n, v_n^n)}^\phi \right) \cdot \left( \tau_{j,i+1}^n \circ 1_{F_1^{v_1^{n-1}, \dots, v_{n-1}^{n-1}}} \right)$$

while the right-hand side is the  $X$ -component of the canonical enhanced 2-isomorphism

$$\tau_{(v_1^n, \dots, v_n^n, i+1), (v_1^{n+1}, \dots, v_{n+1}^{n+1})}^\phi \cdot \left( 1_{F_{n+1}^{i+1}} \circ \tau_{(v_1^{n-1}, \dots, v_{n-1}^{n-1}, j), (v_1^n, \dots, v_{n-1}^n, v_n^n)}^\phi \right).$$

By the coherence Theorem 5.2, both 2-isomorphisms coincide. The second equality is shown in a similar way. Let us now prove that the isomorphism class of the cochain complex is independent of the chosen references. Indeed, suppose we choose other references  $\bar{X}_n, \bar{X}'_n$ , defined by  $n$ -tuples  $(\bar{\mu}_1^n, \dots, \bar{\mu}_n^n)$  and  $(\bar{v}_1^n, \dots, \bar{v}_n^n)$ , for all  $n \geq 1$ . Then, the new  $K$ -module

$$\bar{M}^n = \text{Hom}_{\mathcal{C}^n}(\bar{X}_n, \bar{X}'_n)$$

$n \geq 1$ , is isomorphic to the old one  $M^n$  through the isomorphism  $f^n : M^n \rightarrow \bar{M}^n$  defined by

$$f^n(\varphi) = \left( \tau_{(v_1^n, \dots, v_n^n), (\bar{v}_1^n, \dots, \bar{v}_n^n)}^\phi \right)_X \circ \varphi \circ \left( \tau_{(\bar{\mu}_1^n, \dots, \bar{\mu}_n^n), (\mu_1^n, \dots, \mu_n^n)}^\phi \right)_X.$$

The coboundary operators  $\bar{\delta} : \bar{M}^{n-1} \rightarrow \bar{M}^n$  are defined as before, except that we have to use now the isomorphisms  $\bar{\alpha}_{i,n}^\phi$  and  $\bar{\beta}_{i,n}^\phi$  corresponding to the new references. It easily follows again from Theorem 5.2 that the  $f^n$  define a cochain map. Finally, suppose we choose another object  $Y \cong X$ ,  $Y \in |\mathcal{C}^0|$  and let us denote by  $N^n$  the corresponding  $K$ -modules, namely, for all  $n \geq 1$ ,

$$N^n = \text{Hom}_{\mathcal{C}^n}(Y_n, Y'_n),$$

where

$$Y_n = F_1^{\mu_1^n, \dots, \mu_n^n}(Y),$$

$$Y'_n = F_1^{v_1^n, \dots, v_n^n}(Y).$$

The coboundary maps are defined as before but using the  $Y$ -component of the corresponding canonical enhanced 2-isomorphisms, i.e., the isomorphisms

$$\gamma_{i,n}^\phi \equiv \left( \tau_{(v_1^{n-1}, \dots, v_{n-1}^{n-1}, i), (v_1^n, \dots, v_{n-1}^n, v_n^n)}^\phi \right)_Y : F_n^i(Y'_{n-1}) \rightarrow Y'_n,$$

$$\eta_{i,n}^\phi \equiv \left( \tau_{(\mu_1^{n-1}, \dots, \mu_{n-1}^{n-1}, i), (\mu_1^n, \dots, \mu_{n-1}^n, i)}^\phi \right)_Y : Y_n \rightarrow F_n^i(Y_{n-1})$$

instead of the  $\alpha_{i,n}^\phi$  and  $\beta_{i,n}^\phi$ . Now, if  $h : X \rightarrow Y$  is an isomorphism, it follows immediately from the naturality of the canonical enhanced 2-isomorphisms that

$$F_1^{v_1^n, \dots, v_{n-1}^{n-1}, v_n^n}(h) \circ \alpha_{i,n}^\phi = \gamma_{i,n}^\phi \circ F_1^{v_1^{n-1}, \dots, v_{n-1}^{n-1}, i}(h)$$

and that a similar relation holds between the  $\beta_{i,n}^\phi$  and the  $\eta_{i,n}^\phi$ . Then, defining isomorphisms  $f^n : M^n \rightarrow N^n$  by

$$f^n(\varphi) = F_1^{v_1^n, \dots, v_{n-1}^{n-1}, v_n^n}(h) \circ \varphi \circ F_1^{\mu_1^{n-1}, \dots, \mu_{n-1}^{n-1}, i}(h^{-1}),$$

it is easily checked that we obtain an isomorphism of cochain complexes.  $\square$

**Remark 5.5.** Enhanced 2-cosemisimplicial objects are needed to define cochain complexes with coboundary maps of the form (5.1), where the alternating sum is over all coface functors  $F_n^i$ , for all  $i = 0, \dots, n$ . However, it is well-known that, given a cosemisimplicial object in an abelian category, there are other cochain complexes that may be defined from it. For example, one may define the so-called *path space cochain complex* (see [16]), a cochain complex starting at  $X^1$  instead of at  $X^0$  and whose coboundary maps are given by the alternating sum  $\delta = \delta_n^1 - \delta_n^2 + \dots + (-1)^{n+1} \delta_n^n$ , where the first coface map  $\delta_n^0$  has been omitted. In this sense, it is worth to point out that some of these alternative cochain complexes can be defined even for *arbitrary* 2-cosemisimplicial objects in  $\mathbf{Cat}_K$ . In particular, this is the case for the dual path space cochain complex of the previous path space, which is a cochain complex starting at  $X^2$  and with coboundary maps given by  $\delta = \delta_n^1 - \delta_n^3 + \dots + (-1)^n \delta_n^{n-1}$  (both  $\delta_n^0$  and  $\delta_n^n$  are omitted). We leave to the reader to check that it is indeed possible to choose reference objects  $X_n, X'_n$  in such a way that all the involved domains and codomains of the maps  $F_n^i$ , for all  $i = 1, \dots, n-1$ , belong to the same connected component of the graph  $G_{1,n-1}$ , so that no enhancement is needed in this case to construct a cochain complex by the previous method.

## 6. 2-Cosemisimplicial object of a pseudofunctor and the deformation complex

We are now in a position that enables us to prove the result mentioned in the introduction. Namely, that associated to any pseudofunctor  $\mathcal{F}$  there is a 2-cosemisimplicial object in  $\mathbf{Cat}$  and that, when  $\mathcal{F}$  is  $K$ -linear, the cochain complex  $X^\bullet(\mathcal{F})$  introduced in [6] is the cochain complex obtained by the above method from the corresponding 2-cosemisimplicial object in  $\mathbf{Cat}_K$ .

Let  $\mathcal{F} : \mathfrak{C} \rightarrow \mathfrak{D}$  be an arbitrary pseudofunctor between 2-categories. Included in these data, we have three collections of functors. Namely, the composition functors of  $\mathfrak{C}$  and  $\mathfrak{D}$

$$c_{X,Y,Z}^{\mathfrak{C}} : \mathfrak{C}(X,Y) \times \mathfrak{C}(Y,Z) \rightarrow \mathfrak{C}(X,Z), \quad X,Y,Z \in |\mathfrak{C}|,$$

$$c_{U,V,W}^{\mathfrak{D}} : \mathfrak{D}(U,V) \times \mathfrak{D}(V,W) \rightarrow \mathfrak{D}(U,W), \quad U,V,W \in |\mathfrak{D}|$$

and the functors

$$\mathcal{F}_{X,Y} : \mathfrak{C}(X,Y) \rightarrow \mathfrak{D}(\mathcal{F}(X), \mathcal{F}(Y)), \quad X,Y \in |\mathfrak{C}|$$

defining the action of  $\mathcal{F}$  on the 1- and 2-morphisms. From such functors, and given  $X_0, \dots, X_n \in |\mathfrak{C}|$ , we may construct various iterates, differing in the way they apply an arbitrary path of 1-morphisms in  $\mathfrak{C}$

$$\gamma : X_0 \xrightarrow{f_1} X_1 \rightarrow \dots \rightarrow X_{n-1} \xrightarrow{f_n} X_n$$

to a path in  $\mathfrak{D}$ . More precisely, we define the following notion of iterate of  $\mathcal{F}$ .

**Definition 6.1.** Given  $n \geq 1$  and  $X_0, \dots, X_n \in |\mathfrak{C}|$ , an  $\mathcal{F}_{X_0, \dots, X_n}$ -iterate is any functor

$$H_{X_0, \dots, X_n} : \mathfrak{C}(X_0, X_1) \times \dots \times \mathfrak{C}(X_{n-1}, X_n) \rightarrow \mathfrak{D}(\mathcal{F}(X_0), \mathcal{F}(X_n))$$

obtained as a composite of products of the functors  $\mathcal{F}_{X,Y}$ ,  $c_{X,Y,Z}^{\mathfrak{C}}$ ,  $c_{U,V,W}^{\mathfrak{D}}$ , for all  $X,Y,Z \in \{X_0, \dots, X_n\}$  and  $U,V,W \in \{\mathcal{F}(X_0), \dots, \mathcal{F}(X_n)\}$ , and possibly identity functors. By an  $\mathcal{F}$ -iterate of multiplicity  $n$ , or simply an  $n$ -iterate if there is no ambiguity, we will mean a collection  $H = \{H_{X_0, \dots, X_n}\}_{(X_0, \dots, X_n) \in |\mathfrak{C}|^{n+1}}$ , where  $H_{X_0, \dots, X_n}$  is an  $\mathcal{F}_{X_0, \dots, X_n}$ -iterate, called the  $(X_0, \dots, X_n)$ -component of  $H$ , the same for all collections  $X_0, \dots, X_n$ .

**Remark 6.2.** When  $\mathcal{F}$  is  $K$ -linear, the iterates may be thought of as  $K$ -linear functors from  $\mathfrak{C}(X_0, X_1) \odot \dots \odot \mathfrak{C}(X_{n-1}, X_n)$  to  $\mathfrak{D}(\mathcal{F}(X_0), \mathcal{F}(X_n))$ , where  $\odot$  denotes the Deligne product of  $K$ -linear categories.

According to the previous definition, the image of the above path  $\gamma$  by the  $(X_0, \dots, X_n)$ -component of a generic  $n$ -iterate  $H$  will be of the form

$$\mathcal{F}(f_n \circ \dots \circ f_{i_1 + \dots + i_{r-1} + 1}) \circ \dots \circ \mathcal{F}(f_{i_1 + i_2 + 1} \circ \dots \circ f_{i_1 + 1}) \circ \mathcal{F}(f_{i_1} \circ \dots \circ f_1)$$

for some ordered partition  $\{1, \dots, i_1\}, \{i_1 + 1, \dots, i_1 + i_2\}, \{i_1 + i_2 + 1, \dots, i_1 + i_2 + i_3\}, \dots, \{i_1 + \dots + i_{r-1} + 1, \dots, i_1 + i_2 + \dots + i_r\}$  of the set  $\{1, \dots, n\}$ , with  $i_1 + i_2 + \dots + i_r = n$  and  $1 \leq r \leq n$ . Since such a partition completely defines the corresponding  $n$ -iterate and the partition itself is completely given by the sequence  $(i_1, \dots, i_r)$ , the corresponding  $n$ -iterate will be denoted by  $\mathcal{F}^{(i_1, \dots, i_r)}$ . For example, there is a unique  $\mathcal{F}$ -iterate of multiplicity  $n = 1$ , namely,  $\mathcal{F}^{(1)}$ , given by the family of functors defining the pseudofunctor  $\mathcal{F}$  itself. For  $n = 2$ , we have two different  $\mathcal{F}$ -iterates,  $\mathcal{F}^{(1,1)}$  and  $\mathcal{F}^{(2)}$ , sending the path  $X \xrightarrow{f} Y \xrightarrow{g} Z$  to  $\mathcal{F}(g) \circ \mathcal{F}(f)$  and  $\mathcal{F}(g \circ f)$ , respectively. Their

$(X, Y, Z)$ -components are given by

$$\mathcal{F}_{X,Y,Z}^{(1,1)} = c_{\mathcal{F}(X), \mathcal{F}(Y), \mathcal{F}(Z)}^{\mathfrak{D}} \circ (\mathcal{F}_{X,Y} \times \mathcal{F}_{Y,Z}) \quad X, Y \in |\mathfrak{C}|,$$

$$\mathcal{F}_{X,Y,Z}^{(2)} = \mathcal{F}_{X,Z} \circ c_{\mathcal{F}(X), \mathcal{F}(Y), \mathcal{F}(Z)}^{\mathfrak{D}} \quad X, Y \in |\mathfrak{C}|$$

(in the  $K$ -linear case, the product  $\times$  should be replaced by the Deligne product  $\odot$ ). The reader may easily check that there are four 3-iterates, which are exactly those defined by the families of functors appearing in Lemma 2.2.

**Definition 6.3.** Given two  $n$ -iterates  $H, H'$  of  $\mathcal{F}$ ,  $n \geq 1$ , we will call indexed natural transformation from  $H$  to  $H'$ , and denote it by  $\psi : H \Rightarrow H'$ , any collection of natural transformations between the corresponding components, i.e.,

$$\psi = \{\psi_{X_0, \dots, X_n} : H_{X_0, \dots, X_n} \Rightarrow H'_{X_0, \dots, X_n}\}_{(X_0, \dots, X_n) \in |\mathfrak{C}|^{n+1}}.$$

The natural transformation  $\psi_{X_0, \dots, X_n}$  will be called the  $(X_0, \dots, X_n)$ -component of  $\psi$ .

Notice that, in this definition, no relation is required between the natural transformations corresponding to the various components  $\psi_{X_0, \dots, X_n}$  of  $\psi$ , for different collections of objects  $(X_0, \dots, X_n)$ .

Given two such indexed natural transformations  $\psi : H \Rightarrow H'$  and  $\psi' : H' \Rightarrow H''$ , for some  $n$ -iterates  $H, H', H''$ , we define their vertical composite as the indexed natural transformation  $\psi' \cdot \psi : H \Rightarrow H''$  whose components are given by the usual vertical composition of natural transformations, i.e.,

$$(\psi' \cdot \psi)_{X_0, \dots, X_n} = \psi'_{X_0, \dots, X_n} \cdot \psi_{X_0, \dots, X_n}. \quad (6.1)$$

The 2-cosemisimplicial object of  $\mathcal{F}$  in **Cat** is then defined as follows. Take  $\mathcal{C}^0 = \mathcal{C}^0(\mathcal{F}) = \mathbf{1}$ , the terminal category with only one object and one (identity) morphism. For  $n \geq 1$ , let  $\mathcal{C}^n(\mathcal{F})$  be the small category with objects all  $n$ -iterates of  $\mathcal{F}$  and morphisms the indexed natural transformations between them as defined above, the composition being the above vertical composition. As regards the coface functors, they will be denoted by  $O_n^i : \mathcal{C}^{n-1}(\mathcal{F}) \rightarrow \mathcal{C}^n(\mathcal{F})$ , and they are defined as follows. If  $n = 1$ , both  $O_1^0$  and  $O_1^1$  are equal to the unique possible functor from  $\mathbf{1}$  to  $\mathcal{C}^1(\mathcal{F})$ . If  $n \geq 2$ , let

- $O_n^0$  be the functor sending the  $(n-1)$ -iterate  $H$  to

$$O_n^0(H)_{X_0, \dots, X_n} = c_{\mathcal{F}(X_0), \mathcal{F}(X_1), \mathcal{F}(X_n)}^{\mathfrak{D}} \circ (\mathcal{F}_{X_0, X_1} \times H_{X_1, \dots, X_n})$$

and an indexed natural transformation  $\psi : H \Rightarrow H'$  to

$$O_n^0(\psi)_{X_0, \dots, X_n} = \mathbf{1}_{c_{\mathcal{F}(X_0), \mathcal{F}(X_1), \mathcal{F}(X_n)}^{\mathfrak{D}}} \circ (\mathbf{1}_{\mathcal{F}_{X_0, X_1}} \times \psi_{X_1, \dots, X_n}).$$

- $O_n^i$ , for  $i = 1, \dots, n-1$ , be the functor sending the  $(n-1)$ -iterate  $H$  to

$$O_n^i(H)_{X_0, \dots, X_n} = H_{X_0, \dots, \hat{X}_i, \dots, X_n} \circ (\text{id}_0 \times \dots \times c_{X_{i-1}, X_i, X_{i+1}}^{\mathfrak{C}} \times \dots \times \text{id}_n)$$

and an indexed natural transformation  $\psi : H \Rightarrow H'$  to

$$O_n^i(\psi)_{X_0, \dots, X_n} = \psi_{X_0, \dots, \hat{X}_i, \dots, X_n} \circ (\mathbf{1}_{\text{id}_0} \times \dots \times \mathbf{1}_{c_{X_{i-1}, X_i, X_{i+1}}^{\mathfrak{C}}} \times \dots \times \mathbf{1}_{\text{id}_n})$$

(for short, we write here  $\text{id}_i$  instead of  $\text{id}_{\mathfrak{C}(X_i, X_{i+1})}$ ), and

- $O_n^n$  be the functor sending the  $(n-1)$ -iterate  $H$  to

$$O_n^n(H)_{X_0, \dots, X_n} = c_{\mathcal{F}(X_0), \mathcal{F}(X_{n-1}), \mathcal{F}(X_n)}^{\mathfrak{D}} \circ (H_{X_0, \dots, X_{n-1}} \times \mathcal{F}_{X_{n-1}, X_n})$$

and an indexed natural transformation  $\psi : H \Rightarrow H'$  to

$$O_n^n(\psi)_{X_0, \dots, X_n} = \mathbf{1}_{c_{\mathcal{F}(X_0), \mathcal{F}(X_{n-1}), \mathcal{F}(X_n)}^{\mathfrak{D}}} \circ (\psi_{X_0, \dots, X_{n-1}} \times \mathbf{1}_{\mathcal{F}_{X_{n-1}, X_n}}).$$

The reader may easily check that the above formulas are indeed functorial. Notice also that all these coface functors correspond to all possible ways of getting an  $n$ -iterate from an  $(n-1)$ -iterate.

It is a tedious but straightforward computation to check that these functors  $O_n^i$  satisfy the cosemisimplicial identities (4.1) for all  $0 \leq i < j \leq n+1$  except for the pairs  $i=0, j=1$  and  $i=n, j=n+1$ , with  $n \geq 1$ . When  $n=1$ ,  $O_2^0 \circ O_1^0 : \mathcal{C}^0(\mathcal{F}) \rightarrow \mathcal{C}^2(\mathcal{F})$  is the functor sending the unique object  $\star$  of  $\mathcal{C}^0(\mathcal{F})$  to the 2-iterate  $\mathcal{F}^{(2)}$ , while  $O_2^0 \circ O_1^0$  sends it to  $\mathcal{F}^{(1,1)}$ . Hence, it makes sense to define a natural isomorphism  $\tau_{0,1}^1 : O_2^0 \circ O_1^0 \Rightarrow O_2^0 \circ O_1^0$  whose unique component  $\tau_{0,1}^1(\star)$  is the indexed natural transformation with  $(X, Y, Z)$ -component given by

$$\tau_{0,1}^1(\star)_{X,Y,Z} := \widehat{\mathcal{F}}_{X,Y,Z}^{-1}.$$

Similarly,  $O_2^0 \circ O_1^1$  sends the object  $\star$  to the 2-iterate  $\mathcal{F}^{(1,1)}$  while  $O_2^1 \circ O_1^1$  sends it to  $\mathcal{F}^{(2)}$ , so that we can define  $\tau_{1,2}^1 : O_2^0 \circ O_1^1 \Rightarrow O_2^1 \circ O_1^1$  by

$$\tau_{1,2}^1(\star)_{X,Y,Z} := \widehat{\mathcal{F}}_{X,Y,Z}$$

for all  $X, Y, Z$ . When  $n \geq 2$ , the images of an arbitrary  $(n-1)$ -iterate  $H$  by the functors  $O_{n+1}^1 \circ O_n^0$ ,  $O_{n+1}^0 \circ O_n^0$ ,  $O_{n+1}^{n+1} \circ O_n^n$  and  $O_{n+1}^n \circ O_n^n$  are, respectively, given by

$$\begin{aligned} (O_{n+1}^1 \circ O_n^0)(H)_{X_0, \dots, X_{n+1}} &= c_{\mathcal{F}(X_0), \mathcal{F}(X_2), \mathcal{F}(X_{n+1})}^{\mathfrak{D}} \circ (\mathcal{F}_{X_0, X_1, X_2}^{(2)} \times H_{X_2, \dots, X_{n+1}}), \\ (O_{n+1}^0 \circ O_n^0)(H)_{X_0, \dots, X_{n+1}} &= c_{\mathcal{F}(X_0), \mathcal{F}(X_2), \mathcal{F}(X_{n+1})}^{\mathfrak{D}} \circ (\mathcal{F}_{X_0, X_1, X_2}^{(1,1)} \times H_{X_2, \dots, X_{n+1}}), \\ (O_{n+1}^{n+1} \circ O_n^n)(H)_{X_0, \dots, X_{n+1}} &= c_{\mathcal{F}(X_0), \mathcal{F}(X_{n-1}), \mathcal{F}(X_{n+1})}^{\mathfrak{D}} \circ (H_{X_0, \dots, X_{n-1}} \times \mathcal{F}_{X_{n-1}, X_n, X_{n+1}}^{(1,1)}), \\ (O_{n+1}^n \circ O_n^n)(H)_{X_0, \dots, X_{n+1}} &= c_{\mathcal{F}(X_0), \mathcal{F}(X_{n-1}), \mathcal{F}(X_{n+1})}^{\mathfrak{D}} \circ (H_{X_0, \dots, X_{n-1}} \times \mathcal{F}_{X_{n-1}, X_n, X_{n+1}}^{(2)}). \end{aligned}$$

Hence, for all  $n \geq 2$ , we can define natural isomorphisms  $\tau_{0,1}^n : O_{n+1}^1 \circ O_n^0 \Rightarrow O_{n+1}^0 \circ O_n^0$  and  $\tau_{n,n+1}^n : O_{n+1}^{n+1} \circ O_n^n \Rightarrow O_{n+1}^n \circ O_n^n$  whose  $H$ -components, for any  $(n-1)$ -iterate  $H$ , are the indexed natural transformations with components

$$\tau_{0,1}^n(H)_{X_0, \dots, X_{n+1}} = \mathbf{1}_{c_{\mathcal{F}(X_0), \mathcal{F}(X_2), \mathcal{F}(X_{n+1})}^{\mathfrak{D}}} \circ (\widehat{\mathcal{F}}_{X_0, X_1, X_2}^{-1} \times \mathbf{1}_{H_{X_2, \dots, X_{n+1}}})$$

and

$$\tau_{n,n+1}^n(H)_{X_0, \dots, X_{n+1}} = \mathbf{1}_{c_{\mathcal{F}(X_0), \mathcal{F}(X_{n-1}), \mathcal{F}(X_{n+1})}^{\mathfrak{D}}} \circ (\mathbf{1}_{H_{X_0, \dots, X_{n-1}}} \times \widehat{\mathcal{F}}_{X_{n-1}, X_n, X_{n+1}}).$$

We have then the following:

**Theorem 6.4.** *For any pseudofunctor  $\mathcal{F} : \mathfrak{C} \rightarrow \mathfrak{D}$ , the triple  $(\mathcal{C}^\bullet(\mathcal{F}), O, \tau)$ , with all  $\tau_{ij}^n$ 's equal to identities except in the cases  $(i=0, j=1)$  and  $(i=n, j=n+1)$ , where they are given as above, defines a 2-cosemisimplicial object in **Cat**.*

**Proof.** We have to see that the 2-isomorphisms  $\tau_{ij}^n$ , as defined above, satisfy the coherence relations in Definition 4.1, for all triples  $(i, j, k)$  with  $0 \leq i < j < k \leq n + 2$ . Almost all such conditions are empty because many of the  $\tau$ 's are trivial. It is easy to see that the only nonempty conditions correspond to the triples  $(i, j, k)$  of one of the following two families:

- $i = 0, j = 1$  and  $k \in \{2, \dots, n + 2\}$ , and
- $i \in \{0, \dots, n\}, j = n + 1$  and  $k = n + 2$ .

Let us consider the case  $n = 1$ . In this case, the following four conditions must be checked:

$$\begin{aligned} (1_{O_3^0} \circ \tau_{0,1}^1) \cdot (\tau_{0,2}^2 \circ 1_{O_1^0}) \cdot (1_{O_3^2} \circ \tau_{0,1}^1) &= (\tau_{0,1}^2 \circ 1_{O_1^0}) \cdot (1_{O_3^1} \circ \tau_{0,1}^1) \cdot (\tau_{1,2}^2 \circ 1_{O_1^0}), \\ (1_{O_3^0} \circ \tau_{1,2}^1) \cdot (\tau_{0,3}^2 \circ 1_{O_1^1}) \cdot (1_{O_3^2} \circ \tau_{0,2}^1) &= (\tau_{0,2}^2 \circ 1_{O_1^1}) \cdot (1_{O_3^2} \circ \tau_{0,2}^1) \cdot (\tau_{2,3}^2 \circ 1_{O_1^0}), \\ (1_{O_3^0} \circ \tau_{0,2}^1) \cdot (\tau_{0,3}^2 \circ 1_{O_1^0}) \cdot (1_{O_3^2} \circ \tau_{0,1}^1) &= (\tau_{0,1}^2 \circ 1_{O_1^1}) \cdot (1_{O_3^1} \circ \tau_{0,2}^1) \cdot (\tau_{1,3}^2 \circ 1_{O_1^0}), \\ (1_{O_3^1} \circ \tau_{1,2}^1) \cdot (\tau_{1,3}^2 \circ 1_{O_1^1}) \cdot (1_{O_3^2} \circ \tau_{1,2}^1) &= (\tau_{1,2}^2 \circ 1_{O_1^1}) \cdot (1_{O_3^2} \circ \tau_{1,2}^1) \cdot (\tau_{2,3}^2 \circ 1_{O_1^1}). \end{aligned}$$

Proving any one of these equalities means checking that the  $\star$ -component of both natural transformations (which are some indexed natural transformation) coincide. The reader may easily check that in the first and last cases, the condition one gets is the same, namely

$$\sigma_{X,Y,Z,T}^{24} \cdot \sigma_{X,Y,Z,T}^{12} = \sigma_{X,Y,Z,T}^{34} \cdot \sigma_{X,Y,Z,T}^{13},$$

where the natural transformations  $\sigma_{X,Y,Z,T}^{ij}$  are those defined in Lemma 2.2. Hence, both conditions are equivalent to the composition axiom on  $\mathcal{F}$ . As regards the other two equalities, they turn out to be true for all values of  $\widehat{\mathcal{F}}_{X,Y,Z,T}$ . Indeed, the reader may check that the  $\star$ -component of the left- and right-hand side natural transformations in the second condition are both the indexed natural transformation with components

$$\mathbf{1}_{c_{\mathcal{F}(X), \mathcal{F}(Y), \mathcal{F}(T)}^{\mathcal{D}}} \circ (\mathbf{1}_{\mathcal{F}_{X,Y}} \times \widehat{\mathcal{F}}_{Y,Z,T})$$

while in the third condition both are the indexed natural transformation defined by

$$\mathbf{1}_{c_{\mathcal{F}(X), \mathcal{F}(Z), \mathcal{F}(T)}^{\mathcal{D}}} \circ (\widehat{\mathcal{F}}_{X,Y,Z}^{-1} \times \mathbf{1}_{\mathcal{F}_{Z,T}}).$$

When  $n \geq 2$ , the situation is similar. For the extreme values  $k = 2, n + 2$  it turns out that both conditions reduce to the composition axiom on  $\mathcal{F}$ , while in the cases  $k = 3, \dots, n + 1$  they are always satisfied, for all values of  $\widehat{\mathcal{F}}_{X,Y,Z,T}$ .  $\square$

Suppose now that  $\mathcal{F}$  is  $K$ -linear. The  $K$ -linear structure in the target 2-category  $\mathcal{D}$  naturally induces a  $K$ -module structure on the set of indexed natural transformations between any two iterates so that the categories  $\mathcal{C}^n(\mathcal{F})$  are  $K$ -linear. Furthermore, from the definition of Deligne product of natural transformations between  $K$ -linear functors it immediately follows that all coface functors  $O_n^i$  are also  $K$ -linear. Hence, the corresponding 2-cosemisimplicial object of  $\mathcal{F}$  belongs in this case to  $\mathbf{Cat}_K$ . Furthermore,



it is trivially enhanced, because  $O_1^0 = O_1^1$ , so that the cochain complex construction of the previous section can be applied. Notice that, in this case, we have no choice for the object  $X \in |\mathcal{C}^0|$ , because  $\mathcal{C}^0(\mathcal{F})$  has only one object.

**Proposition 6.5.** *If  $\mathcal{F}$  is  $K$ -linear, its deformation complex  $X^\bullet(\mathcal{F})$  coincides with the cochain complex obtained from the previous 2-cosemisimplicial object by the method described above when we take as reference objects in  $\mathcal{C}^n(\mathcal{F})$ , for  $n \geq 1$ , those defined inductively by Eqs. (5.2)–(5.3).*

**Proof.** It is easy to see that these reference objects indeed correspond to the  $n$ -iterates used in Section 3 to define  $X^\bullet(\mathcal{F})$ , i.e.

$$X_n = \mathcal{F}^{(1, \dots, 1)},$$

$$X'_n = \mathcal{F}^{(n)}.$$

The corresponding  $K$ -modules  $M^n = \text{Hom}_{\mathcal{C}^n(\mathcal{F})}(\mathcal{F}^{(1, \dots, 1)}, \mathcal{F}^{(n)})$  may then be identified with the  $X^n(\mathcal{F})$  defined in Section 3. Moreover, under this identification, the coboundary maps given by Eq. (5.5) exactly correspond to those defined in Section 3 for the  $K$ -modules  $X^\bullet(\mathcal{F})$ , the action of the padding operators corresponding to taking the left and right composites with the canonical isomorphisms  $\alpha_{i,n}^1$  and  $\beta_{i,n}^1$ .  $\square$

## 7. Deviation calculus for an arbitrary $K$ -linear category

A basic question regarding the deformation theory of a  $K$ -linear pseudofunctor which remained open in [6] is that of the higher-order obstructions. To settle down this question, we introduce in this section a generalization to arbitrary  $K$ -linear categories of the deviation calculus introduced by Markl and Stasheff for the category of  $K$ -modules [11] and state the corresponding additivity principle. The 2-cosemisimplicial object of  $\mathcal{F}$  introduced in the previous section turns out to fit quite naturally in the framework of this deviation calculus and allows us to give an easy proof that the higher-order obstructions are indeed cocycles in the deformation complex. The proof is deferred to the next section.

Let us start with the following definition, which generalizes the  $K[[h]]$ -linear extension of a  $K$ -linear category and provides the right setting in which doing a deviation calculus.

**Definition 7.1.** Let  $\mathcal{C}$  be any  $K$ -linear category. Then, we will call deviation extension of  $\mathcal{C}$  any complete  $K[[h]]$ -linear category  $\mathcal{C}_h$  which is  $K[[h]]$ -linear isomorphic to the  $K[[h]]$ -linear extension  $\mathcal{C}[[h]]$  defined above.

Hence, if  $\mathcal{C}_h$  is a deviation category of  $\mathcal{C}$ , we have a bijection between objects  $\varphi : |\mathcal{C}_h| \rightarrow |\mathcal{C}|$  and  $K[[h]]$ -linear isomorphisms  $\mathcal{C}_h(X_h, Y_h) \cong \mathcal{C}(X, Y)[[h]]$  for all

$X_h, Y_h \in |\mathcal{C}_h|$  (where  $X = \varphi(X_h)$  and  $Y = \varphi(Y_h)$ ) such that the composition of morphisms in  $\mathcal{C}_h$  corresponds, after these identifications, to taking the usual “product” of formal power series.

**Example 7.2.** If  $\mathcal{C} = \text{Mod}_K$ , the category of  $K$ -modules, then the full subcategory  $\text{Mod}_{K[[h]]}^0$  of  $\text{Mod}_{K[[h]]}$  with objects the topologically free  $K[[h]]$ -modules is a deviation extension of  $\mathcal{C}$ . This follows from the well-known isomorphisms of  $K[[h]]$ -modules  $\text{Hom}_{K[[h]]}(V[[h]], W[[h]]) \cong (\text{Hom}_K(V, W))[[h]]$ .

This is the example considered by Markl and Stasheff. The example we are interested in this paper is the following.

**Example 7.3.** Let  $\mathcal{A}, \mathcal{B}$  be two  $K$ -linear categories. Then, the functor category  $\text{Fun}_K(\mathcal{A}, \mathcal{B})$  with objects all  $K$ -linear functors  $F : \mathcal{A} \rightarrow \mathcal{B}$  and morphisms the natural transformations with the vertical composition is a  $K$ -linear category. It turns out that a deviation extension of  $\text{Fun}_K(\mathcal{A}, \mathcal{B})$  is given by the full subcategory of the functor category  $\text{Fun}_{K[[h]]}(\mathcal{A}[[h]], \mathcal{B}[[h]])$  with objects all  $K[[h]]$ -linear functors  $F_h : \mathcal{A}[[h]] \rightarrow \mathcal{B}[[h]]$  which are  $K[[h]]$ -linear extensions  $F_h = F[[h]]$  of a  $K$ -linear functor  $F : \mathcal{A} \rightarrow \mathcal{B}$ . Let us denote this subcategory by  $\text{Fun}_{K[[h]]}(\mathcal{A}[[h]], \mathcal{B}[[h]])^0$ . That such a category is a deviation extension of  $\text{Fun}_K(\mathcal{A}, \mathcal{B})$  follows from Lemma 2.7.

Let  $\mathcal{C}_h$  be a deviation category of  $\mathcal{C}$  and let us fix isomorphisms  $\varphi_{X,Y}$  as above. We will identify each morphism in  $\mathcal{C}_h$  with the corresponding formal power series as given by these isomorphisms. Let us then consider a “potentially commutative” diagram in  $\mathcal{C}_h$  of the form

$$\begin{array}{ccc} X_h & \xrightarrow{\alpha_h} & Y_h \\ \gamma_h \downarrow & & \downarrow \beta_h \\ T_h & \xrightarrow{\delta_h} & Z_h \end{array}$$

with  $\alpha_h = \sum_{n \geq 0} \alpha_n h^n$ ,  $\alpha_n \in \mathcal{C}(X, Y)$  for all  $n \geq 0$ , and similarly  $\beta_h$ ,  $\gamma_h$  and  $\delta_h$ . Since the composition of two consecutive morphisms in this diagram is given by the usual product rule between formal power series, the commutativity of the diagram is equivalent to the infinite set of equations

$$\sum_{p+q=m} (\beta_p \circ \alpha_q - \delta_p \circ \gamma_q) = 0, \quad m \geq 0.$$

Hence, it makes sense to talk about the commutativity of such a diagram modulo  $h^{n+1}$  (the equations are satisfied for all  $m \leq n$  but possibly not for  $m = n + 1$ ). Following Markl and Stasheff [11], one may then define the deviation for such a diagram as the first non zero coefficient of the map  $\delta_h \circ \gamma_h - \beta_h \circ \alpha_h$ . More explicitly:

**Definition 7.4.** Suppose that a potentially commutative diagram in  $\mathcal{C}_h$  as above commutes modulo  $h^{n+1}$ , but not modulo  $h^{n+2}$ . Then, the deviation of the diagram is the (unique) morphism  $\Psi : X \rightarrow Z$  in  $\mathcal{C}$  determined by the equation

$$\delta_h \circ \gamma_h - \beta_h \circ \alpha_h = \Psi h^{n+1} \bmod h^{n+2}.$$

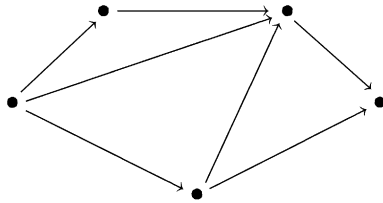
**Remark 7.5.** A priori, the deviation as defined here may depend on the isomorphisms  $\varphi_{X,Y}$  giving  $\mathcal{C}_h$  the structure of a deviation extension of  $\mathcal{C}$ . This is the reason by which we need to fix these isomorphisms.

**Example 7.6.** Given a  $K$ -linear pseudofunctor  $\mathcal{F}$ , let  $\mathcal{C} = \text{Fun}_K(\mathcal{C}^2(\mathcal{F}), \mathcal{C}^3(\mathcal{F}))$ , where  $\mathcal{C}^2(\mathcal{F})$  and  $\mathcal{C}^3(\mathcal{F})$  are the categories that appear in the definition of the 2-co-semisimplicial object associated to  $\mathcal{F}$ . This is a  $K$ -linear category of the form considered in Example 7.3 and a diagram in the corresponding deviation extension is precisely the collection of diagrams (3.10) appearing in Lemma 3.2, for all objects  $X, Y, Z, T$ . If such diagrams commute modulo  $h^{n+1}$  but not modulo  $h^{n+2}$ , an easy computation gives that the deviation is the indexed natural transformation  $\Psi$  with components

$$\begin{aligned} \Psi_{X,Y,Z,T} = & \sum_{p+q=n+1} [\widehat{\mathcal{F}}_{X,Z,T}^p \circ \mathbf{1}_{c_{X,Y,Z}^{\mathcal{E}} \odot \text{id}_{\mathcal{C}(Z,T)}}] \cdot [\mathbf{1}_{c_{\mathcal{F}(X),\mathcal{F}(Z),\mathcal{F}(T)}^{\mathcal{D}}} \circ (\widehat{\mathcal{F}}_{X,Y,Z}^q \odot \mathbf{1}_{\mathcal{F}_{Z,T}})] \\ & - \sum_{p+q=n+1} [\widehat{\mathcal{F}}_{X,Y,T}^p \circ \mathbf{1}_{\text{id}_{\mathcal{C}(X,Y)} \odot c_{Y,Z,T}^{\mathcal{E}}}] \cdot [\mathbf{1}_{c_{\mathcal{F}(X),\mathcal{F}(Y),\mathcal{F}(T)}^{\mathcal{D}}} \circ (\mathbf{1}_{\mathcal{F}_{X,Y}} \odot \widehat{\mathcal{F}}_{Y,Z,T}^q)]. \end{aligned} \quad (7.1)$$

Notice that, in the previous definition, one implicitly chooses an order between the two paths in the diagram, and that the same diagram with the reverse order corresponds to the same deviation but with opposite sign. To indicate which deviation one is considering, an arrow is sometimes drawn in the diagram from the first to the second path. In the example above,  $\Psi$  is the deviation from the path  $\sigma^{24}(h) \cdot \sigma^{12}(h)$  to the path  $\sigma^{34}(h) \cdot \sigma^{13}(h)$ . Clearly, the definition may be extended without trouble to the deviation of any potentially commutative diagram of an arbitrary polygonal shape.

The fundamental point in Markl and Stasheff's deviation calculus is an easy *additivity principle* which allows one to compute the deviation of any potentially commutative diagram having the form of a polygonally subdivided diagram such as that below:



In our general context, this principle can be stated as follows:

**Proposition 7.7.** *Let  $\mathcal{C}_h$  be a deviation category of  $\mathcal{C}$  with fixed isomorphisms  $\varphi_{X,Y}$  for all  $X, Y \in |\mathcal{C}|$ , and let us consider two diagrams in  $\mathcal{C}_h$  with a common edge*

$$\begin{array}{ccc} X_h & \xrightarrow{\alpha_h} & Y_h \\ \gamma_h \downarrow & & \downarrow \beta_h \\ T_h & \xrightarrow{\delta_h} & Z_h \end{array} \quad \begin{array}{ccc} Y_h & \xrightarrow{\varepsilon_h} & U_h \\ \beta_h \downarrow & & \downarrow \eta_h \\ Z_h & \xrightarrow{\zeta_h} & V_h \end{array}$$

Suppose that both diagrams commute modulo  $h^{n+1}$  but not modulo  $h^{n+2}$  and let  $\Psi_1 : X \rightarrow Z$ ,  $\Psi_2 : Y \rightarrow V$  denote the corresponding deviations. Then, the composite diagram

$$\begin{array}{ccc} X_h & \xrightarrow{\varepsilon_h \circ \alpha_h} & U_h \\ \gamma_h \downarrow & & \downarrow \eta_h \\ T_h & \xrightarrow{\zeta_h \circ \delta_h} & V_h \end{array}$$

commutes modulo  $h^{n+1}$  but not modulo  $h^{n+2}$  and its deviation  $\Psi : X \rightarrow V$  is given by

$$\Psi = \zeta_0 \circ \Psi_1 + \Psi_2 \circ \alpha_0.$$

**Proof.** The proof is formally the same as in the case  $\mathcal{C} = \text{Mod}_K$  and is left to the reader (see [11]).  $\square$

Note that, when the zero order terms of the maps  $\alpha_h$  and  $\zeta_h$  are identities (in particular,  $Y = X$  and  $V = Z$ ), deviations simply add, suggesting the name “additivity principle” for this result. Using such a basic additivity principle, we can easily get expressions for the deviation of more complex diagrams. For example, the reader may easily check that the deviation of the previous diagram is simply given by the sum of the deviations of each of the three faces.

For our purposes, the relevant result on this deviation calculus is the following obvious fact:

**Basic fact.** *Let  $D_1, D_2$  be two potentially commutative subdivided polygonal diagrams in a deviation extension  $\mathcal{C}_h$  of a  $K$ -linear category  $\mathcal{C}$ , commuting modulo  $h^{n+1}$  and with a common boundary (consequently defining a 2-dimensional polyhedron topologically equivalent to  $S^2$ ). Then, the deviations of both diagrams must coincide.*

## 8. Higher-order obstructions

Let us now consider the question of the obstructions. Our purpose in this section is to prove, using the previous deviation calculus, that the obstruction to the integrability of a purely pseudofunctorial  $n$ th-order deformation of  $\mathcal{F}$  indeed corresponds to a cocycle in the deformation complex. More explicitly, we have the following.

**Theorem 8.1.** *The obstruction to the extension one higher order of a purely pseudofunctorial  $n$ th-order deformation of a  $K$ -linear unitary pseudofunctor  $\mathcal{F}$  is a 3-cocycle in the corresponding deformation complex  $X^\bullet(\mathcal{F})$ . If this obstruction cocycle defines the zero cohomology class in  $H^3(\mathcal{F})$  an extension exists.*

**Proof.** Let  $(\widehat{\mathcal{F}}_h)_{X,Y,Z} = \widehat{\mathcal{F}}_{X,Y,Z} + \widehat{\mathcal{F}}_{X,Y,Z}^1 h + \cdots + \widehat{\mathcal{F}}_{X,Y,Z}^n h^n$  be a purely pseudofunctorial  $n$ th-order deformation of  $\mathcal{F}$ . Given  $\widehat{\mathcal{F}}^{n+1} = \{\widehat{\mathcal{F}}_{X,Y,Z}^{n+1}\}_{X,Y,Z} \in X^2(\mathcal{F})$ , an easy degree computation shows that  $(\widehat{\mathcal{F}}_h)_{X,Y,Z} + \widehat{\mathcal{F}}_{X,Y,Z}^{n+1} h^{n+1}$  defines an  $(n+1)$ -deformation of the same kind if and only if

$$\delta(\widehat{\mathcal{F}}^{n+1}) = \Psi, \quad (8.1)$$

where the obstruction  $\Psi = \{\Psi_{X,Y,Z,T} : \mathcal{F}_{X,Y,Z,T}^{(1,1,1)} \Rightarrow \mathcal{F}_{X,Y,Z,T}^{(3)}\}_{X,Y,Z,T} \in X^3(\mathcal{F})$  is the indexed natural transformation with components

$$\begin{aligned} \Psi_{X,Y,Z,T} = & \sum_{\substack{p+q=n+1 \\ 0 \leq p,q \leq n}} \left[ \widehat{\mathcal{F}}_{X,Z,T}^p \circ \mathbf{1}_{c_{X,Y,Z}^{\mathfrak{C}} \odot \text{id}_{\mathfrak{C}(Z,T)}} \right] \cdot \left[ \mathbf{1}_{c_{\mathcal{F}(X), \mathcal{F}(Z), \mathcal{F}(T)}}^{\mathfrak{D}} \circ (\widehat{\mathcal{F}}_{X,Y,Z}^q \odot \mathbf{1}_{\mathcal{F}_{Z,T}}) \right] \\ & - \sum_{\substack{p+q=n+1 \\ 0 \leq p,q \leq n}} \left[ \widehat{\mathcal{F}}_{X,Y,T}^p \circ \mathbf{1}_{\text{id}_{\mathfrak{C}(X,Y)} \odot c_{Y,Z,T}^{\mathfrak{C}}} \right] \cdot \left[ \mathbf{1}_{c_{\mathcal{F}(X), \mathcal{F}(Y), \mathcal{F}(T)}}^{\mathfrak{D}} \circ (\mathbf{1}_{\mathcal{F}_{X,Y}} \odot \widehat{\mathcal{F}}_{Y,Z,T}^q) \right]. \end{aligned} \quad (8.2)$$

Notice that these are exactly the components of the indexed natural transformation giving the deviation of diagrams (3.2) (see Eq. (7.1)) except that the sums are taken over all  $p+q=n+1$  such that  $0 \leq p,q \leq n$ . Such restrictions are due to the fact that we are now considering the deviation of diagrams (3.2) when the  $\sigma^{ij}(h)$  are those defined by the  $n$ th-order deformation  $\widehat{\mathcal{F}}_h$  (in particular, we indeed have commutativity modulo  $h^{n+1}$ ).

We want to see that  $\delta(\Psi)=0$  (this is the necessary condition for an  $\widehat{\mathcal{F}}^{n+1}$  satisfying Eq. (8.1) to exist). From the definition of  $\delta : X^2(\mathcal{F}) \rightarrow X^3(\mathcal{F})$  as given in Eq. (5.5), we have that

$$\delta(\Psi) = \sum_{i=0}^4 (-1)^i \alpha_{i,4} \cdot O_4^i(\Psi) \cdot \beta_{i,4}, \quad (8.3)$$

where  $\alpha_{i,4}$  and  $\beta_{i,4}$ ,  $i=0,1,2,3,4$ , denote the  $\star$ -components of some canonical enhanced natural isomorphisms. Explicitly,

$$\alpha_{i,4} = (\tau_{(0,1,2,i),(0,1,2,3)})_{\star},$$

$$\beta_{i,4} = (\tau_{(1,2,3,i),(1,2,3,4)})_{\star}$$

(although not made explicit in the  $\alpha$ 's and  $\beta$ 's, recall that we are taking as enhancing isomorphism  $\phi$  the identity natural transformation of  $O_1^0 = O_1^1$ ). Notice that the

composition in this case is denoted by a dot because it corresponds to the vertical composition of indexed natural transformations (see Eq. (6.1)).

To prove that this is indeed the zero indexed natural transformation, let us apply the  $K[[h]]$ -linear extensions of the functors  $O_4^0$ ,  $O_4^1$ ,  $O_4^2$ ,  $O_4^3$  and  $O_4^4$  to diagram (3.3) from which  $\Psi$  is the deviation.<sup>7</sup> We leave to the reader to check that one obtains the following new diagrams.

Action of  $O_4^0[[h]]$ :

$$\begin{array}{ccc} \mathcal{F}_{X,Y,Z,T}^{(1,1,1,1)}[[h]] & \xrightarrow{\sigma_{X,Y,Z,T}^{1,4}(h)} & \mathcal{F}_{X,Y,Z,T}^{(1,2,1)}[[h]] \\ \sigma_{X,Y,Z,T}^{2,4}(h) \downarrow & & \downarrow \sigma_{X,Y,Z,T}^{5,4}(h) \\ \mathcal{F}_{X,Y,Z,T}^{(1,1,2)}[[h]] & \xrightarrow{\sigma_{X,Y,Z,T}^{7,4}(h)} & \mathcal{F}_{X,Y,Z,T}^{(1,3)}[[h]] \end{array} \quad (8.4)$$

Action of  $O_4^1[[h]]$ :

$$\begin{array}{ccc} \mathcal{F}_{X,Y,Z,T}^{(2,1,1)}[[h]] & \xrightarrow{\sigma_{X,Y,Z,T}^{4,4}(h)} & \mathcal{F}_{X,Y,Z,T}^{(3,1)}[[h]] \\ \sigma_{X,Y,Z,T}^{3,4}(h) \downarrow & & \downarrow \sigma_{X,Y,Z,T}^{11,4}(h) \\ \mathcal{F}_{X,Y,Z,T}^{(2,2)}[[h]] & \xrightarrow{\sigma_{X,Y,Z,T}^{10,4}(h)} & \mathcal{F}_{X,Y,Z,T}^{(4)}[[h]] \end{array} \quad (8.5)$$

Action of  $O_4^2[[h]]$ :

$$\begin{array}{ccc} \mathcal{F}_{X,Y,Z,T}^{(1,2,1)}[[h]] & \xrightarrow{\sigma_{X,Y,Z,T}^{6,4}(h)} & \mathcal{F}_{X,Y,Z,T}^{(3,1)}[[h]] \\ \sigma_{X,Y,Z,T}^{5,4}(h) \downarrow & & \downarrow \sigma_{X,Y,Z,T}^{11,4}(h) \\ \mathcal{F}_{X,Y,Z,T}^{(1,3)}[[h]] & \xrightarrow{\sigma_{X,Y,Z,T}^{9,4}(h)} & \mathcal{F}_{X,Y,Z,T}^{(4)}[[h]] \end{array} \quad (8.6)$$

Action of  $O_4^3[[h]]$ :

$$\begin{array}{ccc} \mathcal{F}_{X,Y,Z,T}^{(1,1,2)}[[h]] & \xrightarrow{\sigma_{X,Y,Z,T}^{8,4}(h)} & \mathcal{F}_{X,Y,Z,T}^{(2,2)}[[h]] \\ \sigma_{X,Y,Z,T}^{7,4}(h) \downarrow & & \downarrow \sigma_{X,Y,Z,T}^{10,4}(h) \\ \mathcal{F}_{X,Y,Z,T}^{(1,3)}[[h]] & \xrightarrow{\sigma_{X,Y,Z,T}^{9,4}(h)} & \mathcal{F}_{X,Y,Z,T}^{(4)}[[h]] \end{array} \quad (8.7)$$

<sup>7</sup> Strictly, what we apply to this diagram are not the  $K[[h]]$ -linear extensions of the  $O_4^i$ , because such extensions act on the category  $\mathcal{C}^3(\mathcal{F})[[h]]$ , whose objects are the same as in  $\mathcal{C}^3(\mathcal{F})$ . But we need to consider a category whose objects are the  $K[[h]]$ -linear extensions of the 3-iterates, not the 3-iterates themselves. Anyway, the meaning of these slightly different versions of the  $O_n^i[[h]]$  is obvious.

Action of  $O_4^4[[h]]$ :

$$\begin{array}{ccc}
 \mathcal{F}_{X,Y,Z,T}^{(1,1,1,1)}[[h]] & \xrightarrow{\sigma_{X,Y,Z,T}^{0,4}(h)} & \mathcal{F}_{X,Y,Z,T}^{(2,1,1)}[[h]] \\
 \sigma_{X,Y,Z,T}^{1,4}(h) \downarrow & & \downarrow \sigma_{X,Y,Z,T}^{4,4}(h) \\
 \mathcal{F}_{X,Y,Z,T}^{(1,2,1)}[[h]] & \xrightarrow{\sigma_{X,Y,Z,T}^{6,4}(h)} & \mathcal{F}_{X,Y,Z,T}^{(3,1)}[[h]]
 \end{array} \quad (8.8)$$

where  $\mathcal{F}^{(1,1,1,1)}$ ,  $\mathcal{F}^{(2,1,1)}$ ,  $\mathcal{F}^{(1,2,1)}$ ,  $\mathcal{F}^{(1,3)}$ ,  $\mathcal{F}^{(2,2)}$ ,  $\mathcal{F}^{(3,1)}$  and  $\mathcal{F}^{(4)}$  denote the eight 4-iterates of  $\mathcal{F}$  and the  $\sigma_{X,Y,Z,T,U}^{i,4}(h)$  are formal power series in  $h$  of the form

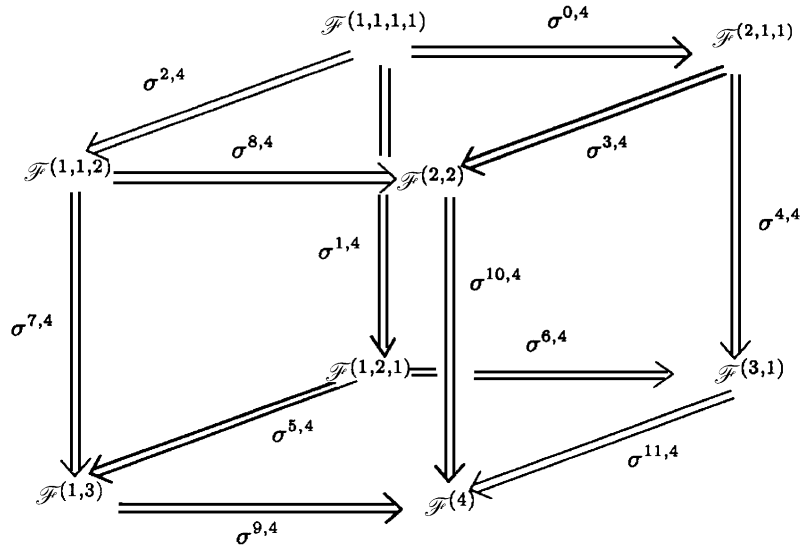
$$\sigma_{X,Y,Z,T,U}^{i,4}(h) = \sum_{k \geq 0} (\sigma_k^{i,4})_{X,Y,Z,T,U} h^k$$

with  $\sigma_k^{i,4} = \{(\sigma_k^{i,4})_{X,Y,Z,T,U}\}_{X,Y,Z,T,U}$ , for  $0 \leq i \leq 11$  and  $k \geq 0$ , the indexed natural transformations with components

$$\begin{aligned}
 (\sigma_k^{0,4})_{X,Y,Z,T,U} &= \mathbf{1}_{c_{\mathcal{F}(X), \mathcal{F}(Z), \mathcal{F}(T), \mathcal{F}(U)}^{\mathcal{D}}} \circ (\widehat{\mathcal{F}}_{X,Y,Z}^k \odot \mathbf{1}_{\mathcal{F}_{Z,T}} \odot \mathbf{1}_{\mathcal{F}_{T,U}}), \\
 (\sigma_k^{1,4})_{X,Y,Z,T,U} &= \mathbf{1}_{c_{\mathcal{F}(X), \mathcal{F}(Y), \mathcal{F}(T), \mathcal{F}(U)}^{\mathcal{D}}} \circ (\mathbf{1}_{\mathcal{F}_{X,Y}} \odot \widehat{\mathcal{F}}_{X,Y,Z}^k \odot \mathbf{1}_{\mathcal{F}_{T,U}}), \\
 (\sigma_k^{2,4})_{X,Y,Z,T,U} &= \mathbf{1}_{c_{\mathcal{F}(X), \mathcal{F}(Y), \mathcal{F}(Z), \mathcal{F}(U)}^{\mathcal{D}}} \circ (\mathbf{1}_{\mathcal{F}_{X,Y}} \odot \mathbf{1}_{\mathcal{F}_{Y,Z}} \odot \widehat{\mathcal{F}}_{Z,T,U}^k), \\
 (\sigma_k^{3,4})_{X,Y,Z,T,U} &= \mathbf{1}_{c_{\mathcal{F}(X), \mathcal{F}(Z), \mathcal{F}(U)}^{\mathcal{D}}} \circ (\mathbf{1}_{\mathcal{F}_{X,Z}} \odot \widehat{\mathcal{F}}_{Z,T,U}^k) \circ \mathbf{1}_{c_{X,Y,Z}^{\mathcal{E}} \odot \text{id}_{Z,T,U}}, \\
 (\sigma_k^{4,4})_{X,Y,Z,T,U} &= \mathbf{1}_{c_{\mathcal{F}(X), \mathcal{F}(T), \mathcal{F}(U)}^{\mathcal{D}}} \circ (\widehat{\mathcal{F}}_{X,Z,T}^k \odot \mathbf{1}_{\mathcal{F}_{T,U}}) \circ \mathbf{1}_{c_{X,Y,Z}^{\mathcal{E}} \odot \text{id}_{Z,T,U}}, \\
 (\sigma_k^{5,4})_{X,Y,Z,T,U} &= \mathbf{1}_{c_{\mathcal{F}(X), \mathcal{F}(Y), \mathcal{F}(U)}^{\mathcal{D}}} \circ (\mathbf{1}_{\mathcal{F}_{X,Y}} \odot \widehat{\mathcal{F}}_{Y,T,U}^k) \circ \mathbf{1}_{\text{id}_{X,Y} \odot c_{Y,Z,T}^{\mathcal{E}} \odot \text{id}_{T,U}}, \\
 (\sigma_k^{6,4})_{X,Y,Z,T,U} &= \mathbf{1}_{c_{\mathcal{F}(X), \mathcal{F}(T), \mathcal{F}(U)}^{\mathcal{D}}} \circ (\widehat{\mathcal{F}}_{X,Y,T}^k \odot \mathbf{1}_{\mathcal{F}_{T,U}}) \circ \mathbf{1}_{\text{id}_{X,Y} \odot c_{Y,Z,T}^{\mathcal{E}} \odot \text{id}_{T,U}}, \\
 (\sigma_h^{7,4})_{X,Y,Z,T,U} &= \mathbf{1}_{c_{\mathcal{F}(X), \mathcal{F}(Y), \mathcal{F}(U)}^{\mathcal{D}}} \circ (\mathbf{1}_{\mathcal{F}_{X,Y}} \odot \widehat{\mathcal{F}}_{Y,Z,U}^k) \circ \mathbf{1}_{\text{id}_{X,Y,Z} \odot c_{Y,Z,T}^{\mathcal{E}}}, \\
 (\sigma_k^{8,4})_{X,Y,Z,T,U} &= \mathbf{1}_{c_{\mathcal{F}(X), \mathcal{F}(Z), \mathcal{F}(U)}^{\mathcal{D}}} \circ (\widehat{\mathcal{F}}_{X,Y,Z}^k \odot \mathbf{1}_{\mathcal{F}_{Z,U}}) \circ \mathbf{1}_{\text{id}_{X,Y,Z} \odot c_{Y,Z,T}^{\mathcal{E}}}, \\
 (\sigma_k^{9,4})_{X,Y,Z,T,U} &= \widehat{\mathcal{F}}_{X,Y,U}^k \circ \mathbf{1}_{\text{id}_{\mathcal{E}(X,Y)} \odot c_{Y,Z,T,U}^{\mathcal{E}}}, \\
 (\sigma_k^{10,4})_{X,Y,Z,T,U} &= \widehat{\mathcal{F}}_{X,Z,U}^k \circ \mathbf{1}_{c_{X,Y,Z}^{\mathcal{E}} \odot c_{Z,T,U}^{\mathcal{E}}}, \\
 (\sigma_k^{11,4})_{X,Y,Z,T,U} &= \widehat{\mathcal{F}}_{X,T,U}^k \circ \mathbf{1}_{c_{X,Y,Z,T}^{\mathcal{E}} \odot \text{id}_{\mathcal{E}(T,U)}}.
 \end{aligned}$$

These diagrams are five of the six faces of the cube in Fig. 5 (for short, when naming the vertices and edges in this diagram, the indexing objects and the formal parameter  $h$  have been omitted).

As regards the lacking face at the top, it turns out to be always commutative (hence, it has null deviation). Indeed, the reader may easily check that, for any path of 1-morphisms  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{l} T, \xrightarrow{m} U$  in  $\mathcal{C}$ , the  $(f, g, l, m)$ -component of the degree  $n$

Fig. 5. Action of the functors  $O_4^i$ ,  $i = 0, 1, 2, 3, 4$  on diagram (3.3).

term in the formal power series giving the composite  $\sigma_{X,Y,Z,T,U}^{3,4}(h) \cdot \sigma_{X,Y,Z,T,U}^{0,4}(h)$  is the 2-morphism

$$\sum_{p+q=n} (\widehat{\mathcal{F}}^p(l, m) \circ 1_{\mathcal{F}(g \circ f)}) \cdot (1_{\mathcal{F}(m) \circ \mathcal{F}(l)} \circ \widehat{\mathcal{F}}^q(f, g))$$

while the same component of the same term for  $\sigma_{X,Y,Z,T,U}^{8,4}(h) \cdot \sigma_{X,Y,Z,T,U}^{2,4}(h)$  is

$$\sum_{p+q=n} (1_{\mathcal{F}(m \circ l)} \circ \widehat{\mathcal{F}}^q(f, g)) \cdot (\widehat{\mathcal{F}}^p(l, m) \circ 1_{\mathcal{F}(g) \circ \mathcal{F}(f)}).$$

By the interchange law, however, both 2-morphisms coincide with  $\widehat{\mathcal{F}}^p(l, m) \circ \widehat{\mathcal{F}}^q(f, g)$ , so that both composites are equal. Hence, the above diagrams nicely fit in a 3-dimensional diagram  $\mathbf{D}$  topologically equivalent to  $S^2$  and to which the basic fact from Section 7 may be applied. Looking at this diagram, it can clearly be subdivided into the two hexagonal diagrams  $D_1$  and  $D_2$  depicted in Fig. 6, whose common boundary is indicated by bold arrows in Fig. 5. Using now the additivity principle (Proposition 7.7), one obtains for the deviation of  $D_1$  the indexed natural transformation

$$\text{Dev}(D_1) = \sigma_0^{11,4} \cdot O_4^4(\Psi) - O_1^4(\Psi) \cdot \sigma_0^{0,4} + O_2^4(\Psi) \cdot \sigma_0^{1,4}$$

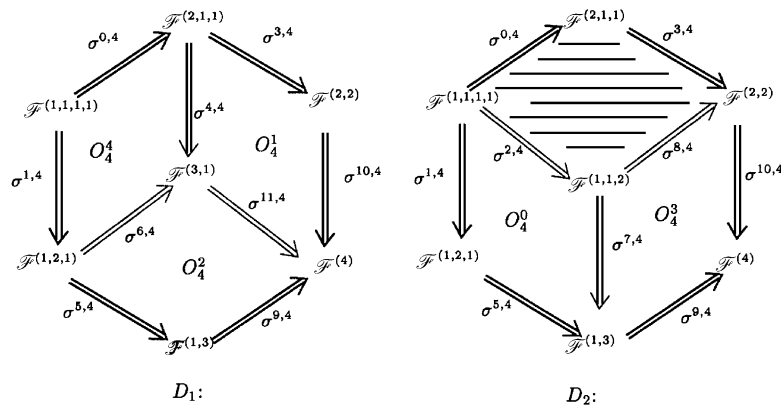
while the deviation of  $D_2$  turns out to be

$$\text{Dev}(D_2) = -\sigma_0^{9,4} \cdot O_0^4(\Psi) + O_3^4(\Psi) \cdot \sigma_0^{2,4}.$$

By the basic fact in the previous section, we know that

$$\text{Dev}(D_1) = \text{Dev}(D_2).$$



Fig. 6. Diagrams  $D_1$  and  $D_2$  decomposing the cube in Fig. 5.

We leave to the reader to check that this is exactly the condition  $\delta(\Psi)=0$ . Notice that taking the composites with the terms  $\sigma_0^{i,4}$  in the above expressions for the deviations of  $D_1$  and  $D_2$ , as established in the additivity principle, corresponds to taking the composites with the  $\alpha_{i,4}$ 's and  $\beta_{i,4}$ 's in Eq. (8.3) and hence to the action of the padding operators.  $\square$

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